



New Classes of Analytic functions Involving Generalized Linear Operator

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Abstract: New subclasses of analytic functions, containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier transformation have been established. Sharp results concerning coefficients, distortion theorems of functions belonging to these classes are determined. Furthermore, Functions with negative coefficients belonging to these classes are also investigated.

2010 Mathematics Subject Classification: 30C45, 30A20, 34A40.

Key words and phrases: Analytic function, linear operator, multiplier transformation.

I. Introduction.

Denote by U the open unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U . Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

In [14], S R Swamy has introduced the following new generalized multiplier differential operator (See [12] also).

Definition 1.1 Let $m \in N_0 = N \cup \{0\}$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I_{\alpha, \beta, \gamma}^m$ was defined by

$$I_{\alpha, \beta, \gamma}^0 f(z) = f(z), I_{\alpha, \beta, \gamma}^1 f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z^2 f''(z)}{\alpha + \beta}, \dots, I_{\alpha, \beta, \gamma}^m f(z) = I_{\alpha, \beta, \gamma}(I_{\alpha, \beta, \gamma}^{m-1} f(z)).$$

Remark 1.2 Observe that for $f(z)$ given by (1.1), we have

$$I_{\alpha, \beta, \gamma}^m f(z) = z + \sum_{k=2}^{\infty} \Phi_k(\alpha, \beta, \gamma, m) a_k z^k, \quad (1.2)$$

where

$$\Phi_k(\alpha, \beta, \gamma, m) = \left(\frac{\alpha + k\beta + k(k-1)\gamma}{\alpha + \beta} \right)^m. \quad (1.3)$$

We note that: i) $I_{\alpha, \beta, 0}^m f(z) = I_{\alpha, \beta}^m f(z)$ ([13]) ii) $I_{1-\beta, \beta, 0}^m f(z) = D_{\beta}^m f(z)$, $\beta \geq 0$ ([1]), iii) $I_{l+1-\beta, \beta, 0}^m f(z) = I_{l, \beta}^m f(z)$, $l > -1$, $\beta \geq 0$ ([3] and it has been considered for $l \geq 0$) and iv) $I_{1-\lambda+\mu, \lambda-\mu, \lambda\mu}^m f(z) = D_{\lambda, \mu}^m f(z)$, $\lambda > (\mu/(\mu+1))$, $\mu \geq 0$ ([5] and they have examined for $\lambda \geq \mu \geq 0$).

Remark 1.3 i) $I_{1-\lambda+\mu, \lambda-\mu, \lambda\mu}^m f(z) = D_{\lambda, \mu}^m f(z)$, $\lambda \geq \mu \geq 0$, was also studied by Raducanu in [6]. ii) $D_1^m f(z)$ was introduced by Salagean [8] and was considered for $m \geq 0$ by Bhoosnurmath and Swamy in [2].

Definition 1.4 ([7]) For $m \in N_0$, $f \in A$, the operator R^m is defined by $R^m : A \rightarrow A$, $R^0 f(z) = f(z)$, $R^1 f(z) = zf'(z)$, ..., $(m+1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z)$, $z \in U$.

Remark 1.5 If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then $R^m f(z) = z + \sum_{k=2}^{\infty} \Omega_k(m) a_k z^k$, $z \in U$, where

$$\Omega_k(m) = \frac{(m+k-1)!}{m!(k-1)!}. \quad (1.4)$$

We now state the following new operator:

Definition 1.6 Let $f \in A$, $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that

$\alpha + \beta > 0$. Denote by $RI_{\alpha, \beta, \gamma, \delta}^m$, the operator given by $RI_{\alpha, \beta, \gamma, \delta}^m : A \rightarrow A$,

$$RI_{\alpha, \beta, \gamma, \delta}^m f(z) = (1-\delta)R^m f(z) + \delta I_{\alpha, \beta, \gamma}^m f(z), z \in U.$$

Clearly i) $RI_{\alpha, \beta, 0, \delta}^m = RI_{\alpha, \beta, \delta}^m$ [10] and [11], ii) $RI_{\alpha, \beta, \gamma, 0}^m = R^m$ [7] and iii) $RI_{\alpha, \beta, \gamma, 1}^m = I_{\alpha, \beta, \gamma}^m$ [13]

Remark 1.7 If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then from (1.2) and Remark 1.4, we have

$$RI_{\alpha, \beta, \gamma, \delta}^m f(z) = z + \sum_{k=2}^{\infty} \{(1-\delta)\Omega_k(m) + \delta\Phi_k(\alpha, \beta, \gamma, m)\} a_k z^k, z \in U,$$

where $\Phi_k(\alpha, \beta, \gamma, m)$ and $\Omega_k(m)$ are as defined in (1.3) and (1.4), respectively.

Motivated by two papers of Swamy [10] and [11], we now introduce new classes, shown below:

Definition 1.8 Let $f \in A$, $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\rho \in [0, 1)$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathfrak{S}_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\operatorname{Re} \left(\frac{[z^2 (RI_{\alpha, \beta, \gamma, \delta}^m f(z))']']}{2RI_{\alpha, \beta, \gamma, \delta}^m f(z)} \right) > \rho, z \in U. \quad (1.5)$$

Definition 1.9 Let $f \in A$, $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\rho \in [0, 1)$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\operatorname{Re} \left(\frac{2z[z(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']']}{[z(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']']'} \right) > \rho, z \in U. \quad (1.6)$$

Definition 1.10 Let $f \in A$, $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\rho \in [0, 1)$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\operatorname{Re} \left(\left(\frac{z(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']']}{RI_{\alpha, \beta, \gamma, \delta}^m f(z)} \right) \left(1 + \frac{z(RI_{\alpha, \beta, \gamma, \delta}^m f(z))''}{(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']']'} \right) \right) > \rho, z \in U. \quad (1.7)$$

In section 2 we study the characterization properties for the function $f \in A$ to belong to the classes $\mathfrak{S}_{\alpha, \beta, \gamma, \delta}^m(\rho)$, $\ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$ and $\mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho)$ by obtaining the coefficient bounds. Distortion theorems of functions belonging to these classes are obtained in section 3. Analytic functions with negative coefficients belonging to the above classes are considered in section 4.

II. General Properties.

In this section we study the characterization properties following the paper of M. Darus and R. Ibrahim [4]. Throughout this paper, unless otherwise mentioned we shall assume that $\zeta_k(\alpha, \beta, \gamma, \delta, m) = (1-\delta)\Omega_k(m) + \Phi_k(\alpha, \beta, \gamma, m)$, where $\Phi_k(\alpha, \beta, \gamma, m)$ and $\Omega_k(m)$ are as defined in (1.3) and (1.4) respectively.

Theorem 2.1 Let $f \in A$, $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\rho \in [0, 1)$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. i) If

$$\sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho), \quad (2.1)$$

then $f(z) \in \mathfrak{I}_{\alpha, \beta, \gamma, \delta}^m(\rho)$.

ii) If

$$\sum_{k=2}^{\infty} \{(2k^2 - (k+1)\rho)\} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho), \quad (2.2)$$

then $f(z) \in \ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$.

iii) If

$$\sum_{k=2}^{\infty} (k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 1 - \rho, \quad (2.3)$$

then $f(z) \in \mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho)$. The results (2.1), (2.2) and (2.3) are sharp.

Proof. i) It suffices to show that the values of $[z^2(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']' / 2RI_{\alpha, \beta, \gamma, \delta}^m f(z)$ lie in a circle centred at 1 with radius $1 - \rho$. We have

$$\begin{aligned} \left| \frac{[z^2(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']'}{2RI_{\alpha, \beta, \gamma, \delta}^m f(z)} - 1 \right| &= \left| \frac{[z^2(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']' - 2RI_{\alpha, \beta, \gamma, \delta}^m f(z)}{2RI_{\alpha, \beta, \gamma, \delta}^m f(z)} \right| \\ &\leq \frac{\sum_{k=2}^{\infty} (k^2 + k - 2) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k|}{2[1 - \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k|]}. \end{aligned}$$

The last expression is bounded above by $1 - \rho$ if

$$\sum_{k=2}^{\infty} (k^2 + k - 2) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho) \left(1 - \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \right),$$

which is equivalent to (2.1). Hence $\left| \frac{[z^2(RI_{\alpha, \beta, \gamma, \delta}^m f(z))']'}{2RI_{\alpha, \beta, \gamma, \delta}^m f(z)} - 1 \right| < 1 - \rho$, and the theorem is proved.

The proof of the second and third part of the theorem is similar and so omitted.

The assertions (2.1), (2.2) and (2.3) are sharp and extremal functions being

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} \frac{2(1 - \rho)}{(k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U, \\ f(z) &= z + \sum_{k=2}^{\infty} \frac{2(1 - \rho)}{\{(2k^2 - (k+1)\rho)\} \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U, \end{aligned}$$

and

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1 - \rho}{(k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U$$

respectively.

Theorem 2.2 Let the hypothesis of the Theorem 2.1 i), ii) and iii) be satisfied. Then

- i) $|a_k| \leq \frac{2(1 - \rho)}{(k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)}, k \geq 2.$
- ii) $|a_k| \leq \frac{2(1 - \rho)}{\{(2k^2 - (k+1)\rho)\} \zeta_k(\alpha, \beta, \gamma, \delta, m)}, k \geq 2.$
- iii) $|a_k| \leq \frac{1 - \rho}{(k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)}, k \geq 2.$

The following inclusion results can be proved using Theorem 2.1.

Theorem 2.3 Let $0 \leq \rho_1 \leq \rho_2 < 1$. Then

i) $\mathfrak{I}_{\alpha, \beta, \gamma, \delta}^m(\rho_2) \subseteq \mathfrak{I}_{\alpha, \beta, \gamma, \delta}^m(\rho_1)$, ii) $\ell_{\alpha, \beta, \gamma, \delta}^m(\rho_2) \subseteq \ell_{\alpha, \beta, \gamma, \delta}^m(\rho_1)$ and iii) $\mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho_2) \subseteq \mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho_1)$.

III. Distortion theorems

Theorem 3.1 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0, 1), \beta \geq 0, \gamma \geq 0, \alpha$ a real number such that

$\alpha + \beta > 0$. i) If $\sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho)$, then

$$\left| z - \frac{1-\rho}{3-\rho} |z|^2 \right| \leq \left| RI_{\alpha, \beta, \gamma, \delta}^m f(z) \right| \leq \left| z + \frac{1-\rho}{3-\rho} |z|^2 \right|, z \in U.$$

ii) If $\sum_{k=2}^{\infty} [(2k^2 - (k+1)\rho)] \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho)$, then

$$\left| z - \frac{2(1-\rho)}{8-3\rho} |z|^2 \right| \leq \left| RI_{\alpha, \beta, \gamma, \delta}^m f(z) \right| \leq \left| z + \frac{2(1-\rho)}{8-3\rho} |z|^2 \right|, z \in U.$$

iii)) If $\sum_{k=2}^{\infty} (k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 1 - \rho$, then

$$\left| z - \frac{1-\rho}{4-\rho} |z|^2 \right| \leq \left| RI_{\alpha, \beta, \gamma, \delta}^m f(z) \right| \leq \left| z + \frac{1-\rho}{4-\rho} |z|^2 \right|, z \in U.$$

Proof i) Note that

$$(6 - 2\rho) \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq \sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho),$$

by Theorem 2.1. Thus $\sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq \frac{2(1-\rho)}{6-2\rho}$. Hence we obtain

$$\begin{aligned} \left| RI_{\alpha, \beta, \delta}^m f(z) \right| &\leq \left| z + \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| |z|^k \right| \\ &\leq \left| z + |z|^2 \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \right| \\ &\leq \left| z + \frac{1-\rho}{3-\rho} |z|^2 \right|. \end{aligned}$$

Similarly

$$\begin{aligned} \left| RI_{\alpha, \beta, \delta}^m f(z) \right| &\geq \left| z - \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| |z|^k \right| \\ &\geq \left| z - |z|^2 \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \right| \\ &\geq \left| z - \frac{1-\rho}{3-\rho} |z|^2 \right|. \end{aligned}$$

This completes the proof of (i). (ii) and iii) can be proved on similar lines.

Theorem 3.2 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0, 1), \beta \geq 0, \gamma \geq 0, \alpha$ a real number such

that $\alpha + \beta > 0$. i) If $\sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1 - \rho)$, then

$$\left| f(z) \right| \geq \left| z - \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta \Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2 \right|, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

ii) If $\sum_{k=2}^{\infty} [(2k^2 - (k+1)\rho)] \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1-\rho)$, then

$$|f(z)| \geq |z| - \frac{2(1-\rho)}{(8-3\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{2(1-\rho)}{(8-3\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

iii) If $\sum_{k=2}^{\infty} (k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 1 - \rho$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{(4-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(4-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

Proof i) In virtue of Theorem 2.1, we have

$$(6-2\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\} \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) |a_k| \leq 2(1-\rho).$$

Thus $\sum_{k=2}^{\infty} |a_k| \leq \frac{2(1-\rho)}{(6-2\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, \delta)\}}$. So we get

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \leq |z| + \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2.$$

On the other hand

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \geq |z| - \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2.$$

This completes the proof. Proofs of (ii) and (iii) are similar.

IV. Functions with negative coefficients

Let T denote the subclass of A consisting of functions of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$. We denote by $T\mathfrak{S}_{\alpha, \beta, \gamma, \delta}^m(\rho)$, $T\ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$ and $T\mathfrak{R}_{\alpha, \beta, \gamma, \delta}^m(\rho)$, the classes of functions $f(z) \in T$ satisfying (1.5), (1.6) and (1.7) respectively. We study the coefficient estimates, distortion theorems and other properties of these classes, following the paper of H. Silverman [9].

For functions in T , the converses of Theorem 2.1 are also true.

Theorem 4.1 i) A function $f(z) \in T$ is in $T\mathfrak{S}_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\sum_{k=2}^{\infty} (k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k \leq 2(1-\rho). \quad (4.1)$$

ii) A function $f(z) \in T$ is in $T\ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\sum_{k=2}^{\infty} (2k^2 - (k+1)\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k \leq 2(1-\rho). \quad (4.2)$$

iii) A function $f(z) \in T$ is in $T\Re_{\alpha, \beta, \gamma, \delta}^m(\rho)$ if and only if

$$\sum_{k=2}^{\infty} (k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k \leq 1 - \rho. \quad (4.3)$$

The results are sharp.

Proof i) In view of Theorem 2.1, it suffices to prove the only if part. Assume that

$$\operatorname{Re} \left(\frac{[z^2 (RI_{\alpha, \beta, \gamma, \delta}^m f(z))']}{2RI_{\alpha, \beta, \gamma, \delta}^m f(z)} \right) = \operatorname{Re} \left(\frac{2z - \sum_{k=2}^{\infty} k(k+1) \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k z^k}{2z - \sum_{k=2}^{\infty} 2 \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k z^k} \right) > \rho. \quad (4.4)$$

letting $z \rightarrow 1^-$ through real values in (4.4), we obtain

$$2 - \sum_{k=2}^{\infty} k(k+1) \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k \geq 2\rho \left(1 - \sum_{k=2}^{\infty} \zeta_k(\alpha, \beta, \gamma, \delta, m) a_k \right).$$

Hence we obtain (4.1), and the proof is complete. The proof of ii) and iii) are similar.

Finally, we note that assertions (4.1), (4.2) and (4.3) of Theorem 4.1 are sharp and extremal functions are

$$f(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\rho)}{(k^2 + k - 2\rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U,$$

$$f(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\rho)}{\{(2k^2 - (k+1)\rho\} \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U,$$

and

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{(k^2 - \rho) \zeta_k(\alpha, \beta, \gamma, \delta, m)} z^k, z \in U,$$

respectively.

Our coefficient bounds enable us to prove the following.

Theorem 4.2 Let $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$, $\rho \in [0,1)$, $\beta \geq 0$, $\gamma \geq 0$, α a real number such that $\alpha + \beta > 0$. i) If A function $f(z) \in T$ is in $T\Im_{\alpha, \beta, \gamma, \delta}^m(\rho)$ then

$$|f(z)| \geq |z| - \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

ii) If A function $f(z) \in T$ is in $T\ell_{\alpha, \beta, \gamma, \delta}^m(\rho)$ then

$$|f(z)| \geq |z| - \frac{2(1-\rho)}{(8-3\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{2(1-\rho)}{(8-3\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

iii) If A function $f(z) \in T$ is in $T\Re_{\alpha, \beta, \gamma, \delta}^m(\rho)$ then

$$|f(z)| \geq |z| - \frac{1-\rho}{(4-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(4-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} |z|^2, z \in U.$$

Remark 4.3 The bounds of i) and ii) of Theorem 4.2 are sharp since the equalities are attained for the functions

$$f(z) = z - \frac{1-\rho}{(3-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} z^2 \quad (z = \pm r),$$

$$f(z) = z - \frac{2(1-\rho)}{(8-3\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} z^2 \quad (z = \pm r) \text{ and}$$

$$f(z) = z - \frac{1-\rho}{(4-\rho)\{(m+1)(1-\delta) + \delta\Phi_2(\alpha, \beta, \gamma, m)\}} z^2 \quad (z = \pm r),$$

respectively.

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