



LRS Bianchi Type-I Cosmological Models with Variable G and Λ in General Relativity

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Abstract: Einstein's field equations with variable gravitational constant (G) and cosmological constant (Λ) are considered in the presence of perfect fluid for the LRS Bianchi type-I, satisfying barotropic equation of state ($p = \omega\rho, 0 \leq \omega \leq 1$). A variety of solutions are presented. The physical significances of the respective cosmological models are also discussed.

Keywords: LRS Bianchi Type-I Space-time, Barotropic, Variations in G and Λ

I. INTRODUCTION

The study of Einstein's field equations in the presence of perfect fluid has attracted the attention of many workers. Ever since Dirac first considered the possibility of a G variable [1], there have been numerous modifications of general relativity to allow for a variable G , nevertheless these theories have not gained wide acceptance. However, recently [2-13] a modification has been proposed treating G and Λ as non-constants coupling scalars. So it is considered G and Λ as coupling scalars within the Einstein equations. In this way the usual conservation law, $\text{div}(T_{ij}) = 0$, holds. Taking the divergence of the Einstein equations and using the Bianchi identities we obtain an equation that controls the variation of G and Λ . These are the modified field equations that allow taking into account a variable G and Λ . Nevertheless this approach has some drawbacks, for example, it cannot derive from a Hamiltonian, although there are several advantages in the approach.

There are many publications devoted to study the variation of G and Λ in the framework of flat FRW symmetries [2-13] and all this works have been extended to more complicated geometries, like for example LRS (Locally Rotationally Symmetric) Bianchi-I models, which represent the simplest generalization of the flat FRW models see for example [14-17] in the context of perfect fluids and [20-22] in the context of viscous fluids). Bianchi I models are important in the study of anisotropies.

But in our opinion, the problem arises when one try to solve the resulting field equations. It seems that it is unavoidable to make simplifying hypotheses, or to impose ad hoc some particular behavior for some of the quantities of the model, in order to obtain an exact solution to the field equations. Such simplifying hypothesis are made for mathematical reason (in order to reduce the number of unknowns) although are justified from the physical point of view. In order to solve the field equations, we normally assume the cosmic scale functions to be function of time only.

Therefore, in this paper we consider LRS (Locally Rotationally Symmetric) Bianchi type-I which is the simplest generalization of the FRW (Friedmann-Robertson-Walker) model. The space-time filled with perfect fluid distribution throughout, satisfying barotropic equation of state ($p = \omega\rho, 0 \leq \omega \leq 1$). We will try to show that with this approach all the usual simplifying hypotheses may be deduced from a correct mathematical principle.

The paper is divided in the following sections. Section two is devoted to outline all the ingredients as well as the field equations. In section three, we have obtained consequences of Einstein's field equations considering the cosmic scale functions to be the functions of time only. In section four, we have solved the field equations for four different cases and obtained explicit exact solutions. We have also studied some geometrical properties of the solutions in section four and have given concluding remarks in section six.

II. METRIC AND FIELD EQUATIONS

We consider the space-time admitting LRS Bianchi type-I (taking $c=1$) group of motion in the form

$$ds^2 = -dt^2 + A^2(x,t)dx^2 + B^2(x,t)(dy^2 + dz^2), \quad (1)$$

See for example [42 – 47].

We assume that the cosmic matter is represented by the energy momentum tensor of a perfect fluid

$$T_{ij} = (p + \rho)v_i v_j + p g_{ij}, \quad (2)$$

where we assuming the barotropic equation of state:

$$p = \omega \rho, 0 \leq \omega \leq 1, \quad (3)$$

ρ is the energy density of the cosmic matter and p is its pressure, v_i is the four velocity vector such that $v_i v^i = 1$. The Einstein's field equations with varying G and Λ given by

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi G(x, t) T_{ij} + \Lambda(x, t) g_{ij}. \quad (4)$$

With the help of equation (2), equations (4) for the metric (1) in co-moving coordinate system, i.e., $v_i = (0, 0, 0, 1)$ take the following explicit forms

$$-8\pi p G + \Lambda = -\frac{1}{A^2} \left(\frac{B_1}{B} \right)^2 + 2 \frac{B_{44}}{B} + \left(\frac{B_4}{B} \right)^2, \quad (5)$$

$$-8\pi p G + \Lambda = -\frac{1}{A^2} \left(\frac{B_{11}}{B} - \frac{A_1}{A} \frac{B_1}{B} \right) + \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4}{A} \frac{B_4}{B}, \quad (6)$$

$$8\pi \rho G + \Lambda = -\frac{1}{A^2} \left[2 \frac{B_{11}}{B} - 2 \frac{A_1}{A} \frac{B_1}{B} + \left(\frac{B_1}{B} \right)^2 \right] + 2 \frac{A_4}{A} \frac{B_4}{B} + \left(\frac{B_4}{B} \right)^2, \quad (7)$$

and
$$0 = \frac{B_{14}}{B} - \frac{A_4}{A} \frac{B_1}{B}. \quad (8)$$

Here afterwards the suffixed 1 and 4 after a field variable represent partial derivatives with respect to x and t respectively.

An additional equation for variation of G and Λ is obtained by divergence of Einstein tensor, i.e.,

$$\left(R_{ij} - \frac{1}{2} R g_{ij} \right)_{;j} = 0, \text{ which leads to } (8\pi G T_{ij} - \Lambda g_{ij})_{;j} = 0, \text{ then}$$

$$0 = 8\pi \rho G_4 + \Lambda_4 + 8\pi G \left[\rho_4 + (p + \rho) \left(\frac{A_4}{A} + 2 \frac{B_4}{B} \right) \right], \quad (9)$$

and
$$0 = -\Lambda_1 + 8\pi (G_1 p + G p_1). \quad (10)$$

The usual energy conservation equation $T_{i;j}^j = 0$, yields

$$0 = \rho_4 + (p + \rho) \left(\frac{A_4}{A} + 2 \frac{B_4}{B} \right). \quad (11)$$

Equation (9) together with (11) puts G and Λ in some short of coupled field given by

$$0 = \Lambda_4 + 8\pi \rho G_4, \quad (12)$$

from which, it is clear that Λ is a constant whenever G is a constant.

As the field equations are highly nonlinear in nature, in general, it is difficult to obtain the explicit solution of the field equation. We consider the case where the cosmic scale functions are functions of time only.

III. CONSEQUENCE OF FIELD EQUATIONS

Here, the cosmic scale functions are assumed in the forms

$$A = f(t) \text{ and } B = g(t). \quad (13)$$

In this case, equation (8) is identically satisfied. Substitution (13) in (5)-(6), yield

$$\frac{g_{44}}{g} + \left(\frac{g_4}{g} \right)^2 = \frac{f_4}{f} \frac{g_4}{g} + \frac{f_{44}}{f}. \quad (14)$$

In view of (13), one can take

$$g = F(f(t)). \quad (15)$$

With the help of (15), equation (14) can be expressed in the form

$$\left(\frac{F'}{F} - \frac{1}{f}\right)f_{44} + \left[\frac{F''}{F} + \left(\frac{F'}{F}\right)^2 - \frac{F'}{Ff}\right](f_4)^2 = 0. \quad (16)$$

Here the superscript prime indicates the differentiation with respect to the argument. This equation is satisfied for the following cases:

$$\text{Case I:} \quad \frac{F'}{F} - \frac{1}{f} = 0 \quad \text{and} \quad \frac{F''}{F} + \left(\frac{F'}{F}\right)^2 - \frac{F'}{Ff} = 0. \quad (17ab)$$

$$\text{Case II:} \quad f_{44} = 0 \quad \text{and} \quad \frac{F''}{F} + \left(\frac{F'}{F}\right)^2 - \frac{F'}{Ff} = 0. \quad (18ab)$$

$$\text{Case III:} \quad f_4 = 0. \quad (19ab)$$

$$\text{Case IV:} \quad f_4 = 0 \quad \text{and} \quad \frac{F'}{F} - \frac{1}{f} = 0. \quad (20ab)$$

IV. SOLUTION OF FIELD EQUATIONS

In this section we intend to drive explicit exact solutions of the field equations for each of the above cases.

A. CASE I

Using (17a) in equation (17b) and integrating, we obtain

$$F = r_1 f + r_2, \quad (21)$$

where $r_1 (\neq 0)$ and r_2 are constants of integration. Now, from (17a) and (21)

$$r_2 = 0. \quad (22)$$

In view of equation (21) and (22), equation (15), now yields

$$g = r_1 f, \quad (23)$$

which leads to isotropic model.

Thus, in view of (23), the line-element (1) can be written as

$$ds^2 = -dt^2 + f^2 \{dx^2 + r_1(dy^2 + dz^2)\}, \quad (24)$$

which can be transformed into

$$ds^2 = -dt^2 + f^2 \{dX^2 + dY^2 + dZ^2\}. \quad (25)$$

From (25) it is obvious that f is the scale factor of the model under consideration.

The set of equations (5)-(11), now reduces to

$$2 \frac{f_{44}}{f} + \left(\frac{f_4}{f}\right)^2 = -8\pi pG + \Lambda, \quad (26)$$

$$3 \left(\frac{f_4}{f}\right)^2 = -8\pi \rho G + \Lambda, \quad (27)$$

$$\Lambda_4 + 8\pi \left[G_4 \rho + G \left\{ \rho_4 + 3(p + \rho) \frac{f_4}{f} \right\} \right] = 0, \quad (28)$$

$$\text{and} \quad \rho_4 + 3(p + \rho) \frac{f_4}{f} = 0. \quad (29)$$

In view of the equation (3), equation (27) yields

$$\rho = r f^{-3(\omega+1)}, \quad (30)$$

where r is a positive constant. Therefore, from (26), (27) and (28), we get

$$8\pi G = \frac{-2}{r(\omega+1)} f^{3(\omega+1)} \left[\frac{f_{44}}{f} - \left(\frac{f_4}{f}\right)^2 \right], \quad (31)$$

$$\text{and} \quad \Lambda = \frac{2}{(\omega+1)} \left[\frac{f_{44}}{f} - \left(\frac{f_4}{f}\right)^2 \right] + 3 \left(\frac{f_4}{f}\right)^2. \quad (32)$$

We can find that (12) is satisfied identically by (30), (31) and (32). Thus each of ρ (and hence p), G and Λ is expressed in terms of f and its derivatives.

Further, for determination of f we prefer to follow the lines set by Ellis and Madsen [40] and used by us in our earlier paper [41] also. Accordingly, we take the Hubble parameter H in the form

$$H(f) \equiv \frac{\dot{f}}{f} = a(f^{-n} + 1), \quad (33)$$

where $a(>0)$ and $n(>1)$ are constants, which puts the deceleration parameter q in the form

$$q \equiv -\frac{f\ddot{f}}{(\dot{f})^2} = \frac{n}{f^n + 1} - 1. \quad (34)$$

The form (33) has the advantage of allowing both decelerating and accelerating phases of an expanding universe.

Equation (33) yields

$$f^n = e^{na(t+t_0)} - 1, \quad (35)$$

in which t_0 is constant of integration, which we set to be zero without any qualitative loss of generality. Thus f in the model turns out to be

$$f^n = e^{nat} - 1. \quad (36)$$

B. Case II

In this case (18a)-(18b), immediately integrated, yield

$$\dot{f} = r_1 t + r_2, \quad (37)$$

and

$$F = r_3 f^2 + r_4 = \{r_3(r_1 t + r_2)^2 + r_4\}, \quad (38)$$

where $r_1 (\neq 0)$, $r_2, r_3 (> 0)$, $r_4 (< 0)$ are constants of integration.

In view of (36), equation (15) now yields

$$g = (r_3 f^2 + r_4)^{\frac{1}{2}} = \{r_3(r_1 t + r_2)^2 + r_4\}^{\frac{1}{2}}. \quad (39)$$

Consequently the corresponding metric of our solution can be written as

$$ds^2 = -dt^2 + (r_1 t + r_2)^2 dx^2 + \{r_3(r_1 t + r_2)^2 + r_4\}(dy^2 + dz^2), \quad (40)$$

which can be transformed through a proper choice of coordinates to the form

$$ds^2 = -dT_1^2 + T_1^2 dX^2 + (T_1^2 + r)(dY^2 + dZ^2), \quad (41)$$

where $r = \frac{r_4}{r_3 r_1^2} (< 0)$ is a constant.

The set of equations (5) - (11), now reduces to

$$-8\pi pG + \Lambda = \frac{T_1^2 + 2r}{(T_1^2 + r)^2}, \quad (42)$$

$$8\pi \rho G + \Lambda = \frac{3(T_1^2 + r) - r}{(T_1^2 + r)^2}, \quad (43)$$

$$\Lambda_4 + 8\pi \left[G_4 \rho + G \left\{ \rho_4 + (p + \rho) \frac{3T_1^2 + r}{T_1(T_1^2 + r)} \right\} \right] = 0, \quad (44)$$

and

$$\rho_4 + (p + \rho) \frac{3T_1^2 + r}{T_1(T_1^2 + r)} = 0. \quad (45)$$

In view of equation (3), equation (45) yields

$$\rho = \frac{r'}{\{T_1(T_1^2 + r)\}^{(\omega+1)}}, \quad (46)$$

where r' is a positive constant. Therefore, in view of (45), equations (42) - (43) yields

$$G = \frac{1}{4\pi(\omega+1)r'} \frac{T_1^2 \{T_1(T_1^2 + r)\}^{(\omega+1)}}{(T_1^2 + r)^2} \quad (47)$$

and

$$\Lambda = \frac{3(T_1^2 + r) - r}{(T_1^2 + r)^2} - \frac{2}{(\omega+1)} \frac{T_1^2}{(T_1^2 + r)^2} \quad (48)$$

C. CASE III

In this case (19) integrated to yield

$$f = \text{const.} \quad (49)$$

In view of (49), equations (5) – (6) yield

$$g = (h_1 t + h_2)^{\frac{1}{2}}, \quad (50)$$

where $h_1 (\neq 0)$ and h_2 are constant of integration.

Substitution of (49) and (50) in equations (5) – (11) yields

$$-8\pi p G + \Lambda = \frac{-h_1^2}{4(h_1 t + h_2)^2}, \quad (51)$$

$$8\pi \rho G + \Lambda = \frac{h_1^2}{4(h_1 t + h_2)^2}, \quad (52)$$

$$\Lambda_4 + 8\pi \left[G_4 \rho + G \left\{ \rho_4 + (p + \rho) \frac{h_1}{(h_1 t + h_2)} \right\} \right] = 0, \quad (53)$$

and

$$\rho_4 + (p + \rho) \frac{h_1}{(h_1 t + h_2)} = 0. \quad (54)$$

in this case, the metric (1) can be written as

$$ds^2 = -dt^2 + dx^2 + (h_1 t + h_2)(dy^2 + dz^2), \quad (55)$$

this can be transformed through a proper choice of coordinates to the form

$$ds^2 = -dt^2 + dX^2 + T_2(dY^2 + dZ^2). \quad (56)$$

In view of (3) and (56), equation (54) yield

$$\rho = h T_2^{-(\omega+1)} = \frac{h}{T_2^{(\omega+1)}}, \quad (57)$$

where h is a positive constant. Therefore from (51), (52) and (57), we get

$$8\pi G = \frac{1}{2(\omega+1)h} T_2^{(\omega-1)}, \quad (58)$$

$$\Lambda = \frac{(\omega-1)}{4(\omega+1)} \frac{1}{T_2^2}. \quad (59)$$

Here, we find that (53) satisfied identically by (55), (56) and (57).

D. CASE IV

In this case, integrating (21a), one gets

$$f = \text{constant.} \quad (60)$$

In view of (60), (21b) yield

$$F = \text{constant.} \quad (61)$$

Consequently, we obtain

$$A = \text{constant}, \quad (62)$$

and

$$B = \text{constant.} \quad (63)$$

Substitution of (62) – (63) into equations (5) – (11) yields

$$\rho = 0, G = 0 \text{ and } \Lambda = 0.$$

Which concludes that the perfect fluid does not survive in this case and the space- time become Minkowskian.

V. SOME PHYSICAL AND GEOMETRICAL DISCUSSION

In this section, we study the following physical and geometrical properties of the models obtain in the preceding sections.

Corresponding to f given by (36), the model (25) is found to have

$$\text{Shear} = \sigma = 0 \text{ (identically),}$$

$$\text{Expansion scalar} = \theta = v^i_{;i} = 3 \frac{f_4}{f} = 3H = \frac{3ae^{nat}}{e^{nat} - 1},$$

the semi-colon stands for covariant differentiation;

$$\rho = k(e^{nat} - 1)^{-\frac{3(\omega+1)}{n}}, \quad G = \frac{na^2 e^{nat}}{4\pi k(\omega+1)} (e^{nat} - 1)^{\frac{3(\omega+1)}{n}-2},$$

and

$$\Lambda = \frac{3a^2 e^{2nat}}{(e^{nat} - 1)^2} - \frac{2na^2}{(\omega+1)} \cdot \frac{e^{nat}}{(e^{nat} - 1)^2}.$$

We note that scalar expansion θ becomes indefinitely larger (∞) or indefinitely small (3a) according as $t \rightarrow 0$ or $t \rightarrow \infty$. Also, ρ tends to ∞ when $t \rightarrow 0$, and it tends to zero when $t \rightarrow \infty$. So, the model starts with a big-bang at $t = 0$ and continues expanding afterwards till $t = \infty$ with material content vanishing there. Isotropy prevails throughout, for $\sigma/\theta = 0$ identically in the model. The deceleration parameter $q \rightarrow (n-1)$ when $t \rightarrow 0$ where as

it tends to (-1) when $t \rightarrow \infty$. For dust ($\omega = 0$) distribution case, we find, for $n = 3$ and $a^2 = \frac{1}{3}$, that $G \rightarrow \infty$ and

0 where as $\Lambda \rightarrow -\infty$ and 1; respectively when $t \rightarrow 0$ and ∞ . Thus a positive Λ (employing repulsion) and a negative (employing acceleration) for large t indicate late times accelerations (There are significant observational evidence that the expansion of the Universe is undergoing a late time acceleration [28-30]). This is

associated with relative dominance of Λ over G , for $\frac{\Lambda}{G} \rightarrow 12\pi k$ (positive constant) when $t \rightarrow \infty$. Also, we

have

$$q = 0 \text{ if } e^{\sqrt{3}t} = 3 \Rightarrow t = \frac{1}{\sqrt{3}} \log_e 3 \equiv T \text{ (say).}$$

At $t = T$, we get $\Lambda = \frac{3}{4}$ and $G = \frac{3}{8\pi k}$ so that $\frac{\Lambda}{G} = 2\pi k = 1$ if $k = \frac{1}{2\pi}$. But, $\frac{\Lambda}{G}$ being an increasing

function of t , implies $\Lambda > G$ for $t > T$ for this k . So, the acceleration could have started just after $t = T$ when Λ also has started dominating over G . Prior $t = T$, there were deceleration only; and at $t = T$ itself, neither deceleration nor acceleration was there.

In the radiative ($\omega = \frac{1}{3}$) and stiff ($\omega = 1$) distribution, we get the same result, but for different values of

constant n , as $n = 4$ and $n = 6$ respectively.

Further, the matter density ρ , gravitational constant G and cosmological constant Λ for the model (41) are given by equations (46), (47) and (48) respectively, and expansion scalar θ and shear σ^2 are obtain as

$$\theta = \frac{3T_1^2 + r}{T_1(T_1^2 + r)}, \quad \text{and} \quad \sigma^2 = \frac{1}{3} \frac{r^2}{T_1^2(T_1^2 + r)}.$$

The model has point singularity at $T_1 = 0$. We observe that the model starts with ρ, σ and θ all infinite and

became zero at $T_1 = \infty$. Here $\frac{\sigma^2}{\theta^2} \rightarrow 0$ as $T_1 \rightarrow \infty$ which implies that the model approaches isotropy for large

value of T_1 . Initially cosmological term Λ tends to $\frac{2}{K}(-ve)$ as $K < 0$ and G is zero (finite). As $T_1 \rightarrow \infty$, $\Lambda = 0$

and $G = \infty$, therefore, G is increasing with time, gives empty space. The cosmic scenario starts from a big-bang at $T_1 = 0$ and continues until $T_1 = \infty$. The possibility of G increasing with time, atleast in some stages of the development of the universe has been investigated by Abdel-Rehman [23], Chow [24], Levitt [25] and Milne[26].

Further, for the model (56), cosmological parameters are given by

$$\rho = \frac{h}{T_2^{(\omega+1)}}, \theta = \frac{1}{T_2}, \sigma^2 = \frac{1}{12T_2^2}, G = \frac{1}{16\pi(\omega+1)h} \frac{1}{T_2^{(\omega-1)}}, \text{ and } \Lambda = \frac{\omega-1}{4(\omega+1)} \frac{1}{T_2^2}.$$

Thus this case degenerates only stiff ($\omega=1$) fluid model [27]. In this case the reality and energy conditions are

identically satisfied. Here, $\frac{\sigma^2}{\theta^2} = \frac{1}{12}$ being independent of cosmic time implies that the model does not

approach isotropy. As in preceding cases, this model is also expanding in nature, but expansion is decelerating.

It is seen that $T_2 = 0$ represent big-bang singularity. For matter dominated ($\omega=0$) and radiation dominated

$\left(\omega = \frac{1}{3}\right)$ cases, the gravitational constant G and cosmological constant Λ tends to ∞ and $-\infty$ respectively

when $T_2 \rightarrow 0$, but when $T_2 \rightarrow \infty$ both the terms are become zero. Therefore, G decreases with cosmic time,

whereas, Λ increases with time. Thus, at late time ($T_2 \rightarrow \infty$), in the case ($\omega=0$) dust distribution and

$\left(\omega = \frac{1}{3}\right)$ radiative distribution, both Λ and G are finitely positive.

In the fourth case, it is shown that the fluid distribution does not exist and space times become Minkowskian.

VI. CONCLUSION

In this paper, we have presented a class of solutions to Einstein's field equations with variable G and Λ in LRS Bianchi type-I space-time in the presence of a perfect fluid. In some cases, it is observed that G is an increasing function of time. When the universe is required to have expanded from a finite minimum volume, the critical density assumption and conservation of energy-momentum tensor dictate that G increases in a perpetually expanding universe. The possibility of an increasing G has been suggested by several authors. Four LRS Bianchi type-I models have been obtained in Sections 4.1, 4.2, 4.3 and 4.4. Expressions for some important cosmological parameters have been obtained for all the models and the physical behavior of the models is discussed in detail. In all the cases, the models represent a shearing, non-rotating and expanding model of the universe with a big-bang start approaching isotropy at late times in first two cases and anisotropy for the third case. The resulting model has an arbitrary scale factor whose specification in a particular form renders the model allow late time accelerations in first case. In second case, we find G is increasing function of cosmic time whereas Λ is a decreasing function. In the third case, for stiff fluid G and Λ are become constant and for dust and radiative distribution, we find G is decreasing function of cosmic time whereas Λ is an increasing function. The fourth case does not admit any solution for perfect fluid distribution. Finally, the solutions presented in the paper are new and useful for a better understanding of the evolution of the universe in Bianchi type-I space-time with variables G and Λ .

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