



Occurrence of Digit 0 in All Base b Natural Numbers Less Than b^n

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Abstract: Digit 0 is special in many ways. It's occurrence in general base b numbers is analyzed in this work. If all natural numbers m in range $1 \leq m < b^n$ are considered, then the number of occurrences of digit 0, its first occurrence and last occurrence are formulated. All the analysis is further extended to multiple occurrences of 0's. Using base $b = 16$, tables of values in the range as large as till 1 quintillion are provided.

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I. Introduction

Number 0 is very special in many ways. It is the only digit which is not amongst natural numbers

1, 2, 3, ...

but without which the natural number sequence cannot proceed quite ahead irrespective of the base used for representation!

Although use of decimal system with base 10 is most common, having 10 digits, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, any number system with every base $b > 1$ is feasible.

We use the term number to simply mean natural number. For any number $b > 1$, base b numbers m with $1 \leq m < b^n$ are under consideration. The last number b^n , being the one having $n + 1$ number of significant digits in base b , is omitted.

Because leading 0's are having no significance, we have considered only significant 0's; those which follow some initial non-zero digit.

II. Occurrence of Digit 0

In any number system having place value character, 0 has a great importance [1]. In fact, it is this digit which allows number system to represent almost unlimited number of numbers using only finite number of digits! Here, the occurrence of digit 0 is analyzed in all base b numbers less than b^n , for all natural numbers n .

For instance purpose, we choose base $b = 16$. This base is preferred in modern electronic computer systems at back end system level while actually performing the calculations.

Selecting base power ranges $1 - 16^i$, we have gone as high as up to 16^{15} , just above one quintillion, i.e., 10^{18} and calculated the count of occurrences of 0 digit in all base 16 numbers, i.e., hexadecimal numbers, within the ranges. Although the numbers under inspection are hexadecimal, their counts below are given in decimal.

Table I : Number of Hexadecimal Numbers in Various Ranges with Single and Double 0's in Their Digits

Sr. No.	Numbers Range Less Than	Number of Numbers in Base 16 with single 0	Number of Numbers in Base 16 with two 0's
1.	16^1	0	0
2.	16^2	15	0
3.	16^3	465	15
4.	16^4	10,590	690
5.	16^5	213,090	20,940
6.	16^6	4,009,965	527,190
7.	16^7	72,353,715	11,917,815
8.	16^8	1,268,369,340	251,120,940
9.	16^9	21,771,494,340	5,035,183,440
10.	16^{10}	367,761,728,715	97,299,245,940
11.	16^{11}	6,134,265,634,965	1,827,250,417,815
12.	16^{12}	101,281,580,088,090	33,543,021,902,190
13.	16^{13}	1,658,237,634,775,590	604,426,908,620,940
14.	16^{14}	26,958,773,523,447,465	10,724,641,264,089,690
15.	16^{15}	435,659,737,878,916,201	187,828,392,484,792,815

In the first range $1 \leq m < 16^1 = 16$, 0 just can't occur even once. This range contains only 15 numbers which are all digits except 0, from 1 to F.

In the second range $1 \leq m < 16^2 = 256$, single 0 occurs 15 times. Its appearances are in numbers

10, 20, 30, 40, 50, 60, 70, 80, 90, A0, B0, C0, D0, E0 and F0,

all at unit's places. So, first occurrence count comes as ${}^{2-1}C_1 15^{2-1} = 15$.

In the third range, $1 \leq m < 16^3 = 4,096$, single 0 occurs total 465 times. It occurs in earlier 15 numbers

10, 20, 30, ..., F0,

and then additionally in

101, 102, 103, ..., 10F, 201, 202, 203, ..., 20F, ..., F01, F02, F03, ..., F0F,

110, 120, 130, ..., 1F0, 210, 220, 230, ..., 2F0, ..., F10, F20, F30, ..., FF0,

which is ${}^{3-1}C_1 15^{3-1} = 2 \times 15^2 = 450$ times; totaling to $15 + 450 = 465$ times.

In this range, double zeros occur in

100, 200, 300, ..., F00,

which is ${}^{3-1}C_2 15^{3-2} = 15$ times.

In the fourth range, $1 \leq m < 16^4 = 65,536$, single 0 occurs in earlier 465 numbers

10, 20, 30, ..., F0,

101, 102, 103, ..., 10F, 201, 202, 203, ..., 20F, ..., F01, F02, F03, ..., F0F,

110, 120, 130, ..., 1F0, 210, 220, 230, ..., 2F0, ..., F10, F20, F30, ..., FF0,

and then additionally in

1110, 1120, ..., 11F0, 1210, 1220, ..., 12F0, ..., 1F10, 1F20, ..., 1FF0,

2110, 2120, ..., 21F0, 2210, 2220, ..., 22F0, ..., 2F10, 2F20, ..., 2FF0,

⋮

F110, F120, ..., F1F0, F210, F220, ..., F2F0, ..., FF10, FF20, ..., FFF0,

1101, 1102, ..., 110F, 1201, 1202, ..., 120F, ..., 1F01, 1F02, ..., 1F0F,

2101, 2102, ..., 210F, 2201, 2202, ..., 220F, ..., 2F01, 2F02, ..., 2F0F,

⋮

F011, F012, ..., F01F, F021, F022, ..., F02F, ..., F0F1, F0F2, ..., F0FF,

1011, 1012, ..., 101F, 1021, 1022, ..., 102F, ..., 10F1, 10F2, ..., 10FF,

2011, 2012, ..., 201F, 2021, 2022, ..., 202F, ..., 20F1, 20F2, ..., 20FF,

⋮

F011, F012, ..., F01F, F021, F022, ..., F02F, ..., F0F1, F0F2, ..., F0FF.

They give count ${}^{4-1}C_1 15^{4-1} = 3 \times 15^3 = 10,125$, totaling to $465 + 10,125 = 10,590$

Now in this range double 0 occurs in earlier 9 numbers

100, 200, 300, ..., F00,

and then additionally in

1100, 1200, ..., 1F00, 2100, 2200, ..., 2F00, ..., F100, F200, ..., FF00,

1010, 1020, ..., 10F0, 2010, 2020, ..., 20F0, ..., F010, F020, ..., F0F0,

1001, 1002, ..., 100F, 2001, 2002, ..., 200F, ..., F001, F002, ..., F00F.

They give count ${}^{4-1}C_2 15^{4-2} = 3 \times 15^2 = 675$, totaling to $15 + 675 = 690$.

This way all numbers in above table can be explained.

There is exactly like explanation for count of multiple 0's in hexadecimal numbers within these ranges. In fact, this count fits in a decent formula.

Notation : We propose the notation ${}_b^A O_r^n$ for number of base b numbers less than b^n with r number of 0's.

Theorem 1 : If r , n and b are positive integers with $1 < r < n$ and $b > 1$, then the number of base b numbers containing exactly r number of significant digit 0's in the range $1 \leq m < b^n$ is

$${}_b^A O_r^n = \sum_{j=r+1}^n {}^{j-1}C_r (b-1)^{j-r}.$$

Proof. Let n , r and b be positive integers with $1 < r < n$ and $b > 1$. Base b will have b number of digits. If $b \leq 10$, these digits are denoted by 0, 1, 2, ..., $b-1$. In case $b > 10$, we denote first 10 digits by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and remaining successive digits by alphabets A, B, C, ..., X where X is not exactly alphabet X but merely a symbol to designate that alphabet which at number equal to 9 more than its own number in the list of alphabets.

We prove the result by induction on n and r . The cases of initial values of r are already discussed, wherein the formula is clearly seen to be valid. For any positive r , the minimum value of n is $r+1$. In this case, there is only one possible position for all 0's, viz., last succession, and then there are exactly $(b-1)$ occurrences of r 0's :

$$\underbrace{1000\dots 0}_{r \text{ times}}, \underbrace{2000\dots 0}_{r \text{ times}}, \dots, X \underbrace{000\dots 0}_{r \text{ times}}$$

Also

$${}_b^A O_r^{n=r+1} = \sum_{j=r+1}^n {}^{j-1}C_r (b-1)^{j-r} = {}^{r+1-1}C_r (b-1)^{r+1-r} = {}^r C_r (b-1)^1 = (b-1)$$

So, the formula is true in this case. The block occurrences begin with count $(b-1)$.

The next value of n is $r+2$. Now except the leading position, there are $n-1$, in this case $r+1$, possible positions that can be occupied by r number of 0's. These combinations are ${}^{r+1}C_r$. We explicitly mention these in this case:

$$\begin{aligned} & \underbrace{1000\cdots 0}_{r \text{ times}}, \underbrace{2000\cdots 0}_{r \text{ times}}, \cdots, \underbrace{X000\cdots 0}_{r \text{ times}}, \quad (=b-1) \\ & \underbrace{11000\cdots 0}_{r \text{ times}}, \cdots, \underbrace{1X000\cdots 0}_{r \text{ times}}, \underbrace{21000\cdots 0}_{r \text{ times}}, \cdots, \underbrace{2X000\cdots 0}_{r \text{ times}}, \cdots, \underbrace{X1000\cdots 0}_{r \text{ times}}, \cdots, \underbrace{XX000\cdots 0}_{r \text{ times}} (= (b-1) \times (b-1)) \\ & \underbrace{10100\cdots 0}_{r-1 \text{ times}}, \cdots, \underbrace{10X00\cdots 0}_{r-1 \text{ times}}, \underbrace{20100\cdots 0}_{r-1 \text{ times}}, \cdots, \underbrace{20X00\cdots 0}_{r-1 \text{ times}}, \cdots, \underbrace{X0100\cdots 0}_{r-1 \text{ times}}, \cdots, \underbrace{X0X00\cdots 0}_{r-1 \text{ times}} (= (b-1) \times (b-1)) \\ & \vdots \\ & \underbrace{1000\cdots 01}_{r \text{ times}}, \cdots, \underbrace{1000\cdots 0X}_{r \text{ times}}, \underbrace{2000\cdots 01}_{r \text{ times}}, \cdots, \underbrace{2000\cdots 0X}_{r \text{ times}}, \cdots, \underbrace{X000\cdots 01}_{r \text{ times}}, \cdots, \underbrace{X000\cdots 0X}_{r \text{ times}} (= (b-1) \times (b-1)) \end{aligned}$$

In addition to earlier $(b-1)$ occurrences of r 0's in the first row, now there are $(b-1)^2$ additional occurrences each in ${}^{r+1}C_r$ rows of new possible occurrences and so total occurrences of r 0's is given by

Previous + ${}^{r+1}C_r (b-1)^{n-r}$ and hence equals

$$(b-1) + {}^{r+1}C_r (b-1)^{n-r} = {}^r C_r (b-1) + {}^{r+1}C_r (b-1)^2 = \sum_{j=r+1}^{r+2} {}^{j-1}C_r (b-1)^{j-r} = {}_b^A O_r^{n=r+2}$$

asserting the formula is true in this case.

For the next value of n , i.e., $r+3$, the same scenario will occur. With new availability figure of $r+2$ (except the leading) positions for r 0's, there will be addition of ${}^{r+2}C_r (b-1)^3$ and hence equals

$$\begin{aligned} {}_b^A O_r^{r+2} + {}^{r+2}C_r (b-1)^{4-1} &= \sum_{j=r+1}^{r+2} {}^{j-1}C_r (b-1)^{j-r} + {}^{r+2}C_r (b-1)^{n-r} \\ &= \sum_{j=r+1}^{r+3} {}^{j-1}C_r (b-1)^{j-r} \\ &= {}_b^A O_r^{n=r+3} \end{aligned}$$

Continuing this indefinitely, the formula is proved for all positive integers r and $n > r$.

The table given above is now constructed to higher occurrences of 0's in numbers base 16 by using this formula.

Table II : Number of Hexadecimal Numbers in Various Ranges with Multiple 0's in Their Digits

Sr. No.	Number Range <	Number of Numbers in Base 16 with 3 0's	Number of Numbers in Base 16 with 4 0's	Number of Numbers in Base 16 with 5 0's
1.	16^4	15	0	0
2.	16^5	915	15	0
3.	16^6	34,665	1,140	15
4.	16^7	1,047,165	51,765	1,365
5.	16^8	27,625,290	1,823,640	72,240
6.	16^9	665,500,290	54,979,890	2,907,240
7.	16^{10}	15,017,687,790	1,490,198,640	98,588,490
8.	16^{11}	322,564,562,790	37,370,667,390	2,969,025,990
9.	16^{12}	6,665,718,859,665	883,124,573,640	81,906,057,240
10.	16^{13}	133,528,804,797,165	19,912,587,464,265	2,111,715,432,240
11.	16^{14}	2,607,358,980,578,415	432,217,616,761,140	51,588,318,947,865
12.	16^{15}	49,835,025,972,765,915	9,090,623,231,995,515	1,206,042,400,979,115

Table II : Continued ...

Sr. No.	Number Range <	Number of Numbers in Base 16 with 6 0's	Number of Numbers in Base 16 with 7 0's	Number of Numbers in Base 16 with 8 0's
1.	16^7	15	0	0
2.	16^8	1,590	15	0
3.	16^9	96,090	1,815	15
4.	16^{10}	4,348,590	123,315	2,040
5.	16^{11}	163,817,340	6,198,315	153,915
6.	16^{12}	5,426,286,090	256,792,065	8,507,040
7.	16^{13}	163,300,348,590	9,278,167,065	384,397,665
8.	16^{14}	4,561,220,661,090	302,472,854,565	15,044,132,040
9.	16^{15}	120,006,628,864,215	9,098,313,479,565	528,134,835,165

Table II : Continued ...

Sr. No.	Number Range <	Number of Numbers in Base 16 with 9 0's	Number of Numbers in Base 16 with 10 0's	Number of Numbers in Base 16 with 11 0's
1.	16^{10}	15	0	0
2.	16^{11}	2,265	15	0
3.	16^{12}	187,890	2,490	15
4.	16^{13}	11,325,390	225,240	2,715
5.	16^{14}	554,278,515	14,703,990	265,965
6.	16^{15}	23,358,309,765	774,838,365	18,693,465

Table II : Continued ...

Sr. No.	Number Range <	Number of Numbers in base 16 with 12 0's	Number of Numbers in base 16 with 13 0's	Number of Numbers in base 16 with 14 0's
1.	16^{13}	15	0	0
2.	16^{14}	2,940	15	0
3.	16^{15}	310,065	3,165	15

III. First Occurrence of Digit 0

The first number containing 0 is 10, but this 10 is in the corresponding base in which we are studying our numbers. 10 is first container of significant 0 for all sufficiently high ranges of numbers ($\geq b^1$). For 2 0's, the first instance is 100, for 3 it is 1000 and so on. All these representations are in corresponding bases.

Formula 1 : If n , r and b are natural numbers with $b > 1$, then the first occurrence of r number of 0's in base b numbers in range $1 \leq m < b^n$ is

$$f = \begin{cases} - , & \text{if } r \geq n \\ b^r , & \text{if } r < n \end{cases} .$$

IV. Last Occurrence of Digit 0

For base $b = 16$ of hexadecimal numbers, the following table shows the last number in ranges containing exactly required number of 0's.

Table III : Last Hexadecimal Numbers with Multiple 0's in their Digits in Various Base Power Ranges

Sr. No.	Number Range < → Last Number with ↓	16^1	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9
1.	1 0	-	F0	FF0	F,FF0	FF,FF0	FFF,FF0	F,FFF,FF0	FF,FFF,FF0	FFF,FFF,FF0
2.	2 0's	-	-	F00	F,F00	FF,F00	FFF,F00	F,FFF,F00	FF,FFF,F00	FFF,FFF,F00
3.	3 0's	-	-	-	F,000	FF,000	FFF,000	F,FFF,000	FF,FFF,000	FFF,FFF,000
4.	4 0's	-	-	-	-	F0,000	FF0,000	F,FF0,000	FF,FF0,000	FFF,FF0,000
5.	5 0's	-	-	-	-	-	F00,000	F,F00,000	FF,F00,000	FFF,F00,000
6.	6 0's	-	-	-	-	-	-	F,000,000	FF,000,000	FFF,000,000
7.	7 0's	-	-	-	-	-	-	-	F0,000,000	FF0,000,000
8.	8 0's	-	-	-	-	-	-	-	-	F00,000,000

Their formulation also follows.

Formula 2 : If n , r and b are natural numbers with $b > 1$, then the last occurrence of r number of 0's in base b numbers in range $1 \leq m < b^n$ is

$$l = \begin{cases} - , & \text{if } r \geq n \\ b^n - b^r , & \text{if } r < n \end{cases} .$$

Remark : All the work here has generalized the earlier results in [5].

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