



Combining Z-Valuations

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Abstract: The concepts of Z- number and Z-valuation were introduced by Zadeh. Yager had outlined methods for combining multiple Z- valuations. In this paper we analyze and compare two of these methods. Usually exact methods lead to very complicated calculations. Hence we also introduce an approximation method in the case where the underlying probability distribution is dependent on a single parameter.

Keywords: Fuzzy event probability, Zadeh's Z-number, Z-valuation .

I. Introduction

A Z- number is an ordered pair of fuzzy numbers (A,B).It can be used to describe a random variable say X. If we say X is (A,B) then we have a Z-valuation. It is to be understood as "The probability of the fuzzy event $X=A$ is B". This concept was introduced by Zadeh in 2011[1]. Yager [2] had outlined methods for combining multiple Z- valuations. In this paper we analyze and compare two of these methods.

Usually exact methods lead to very complicated calculations. Hence we also introduce a very useful approximation method in the case where the underlying probability distribution is dependent on a single parameter.

II. Preliminary Definitions

Definition 1: Fuzzy sets

A fuzzy set on a domain U is defined by its membership function from U to $[0, 1]$ is meant a function $A: U \rightarrow [0,1]$. 'A' is called the membership function, $A(x)$ is called membership grade of x in A. we also write it as $A = \{ (x, A(x)) : x \in U \}$ [3]

Definition 2: α -cuts of a fuzzy sets

Given a fuzzy set A on U, a domain and a number α in I, such that $0 < \alpha \leq 1$, then associate a crisp set with A, denoted by A_α , defined as $A_\alpha = \{ x \in U / A(x) \geq \alpha \}$ is called the α - cut of A. For each α , we obtain an the α - cut of A. [6]

Definition 3: Fuzzy event probability

If X is a random variable, then X is A represents a fuzzy event in R, the real line. The fuzzy event probability of "X is A" [7] (ie) probability measure of this event, may be expressed as (Zadeh 1968) $FEP(X$

$\in A) = \int_R \mu_A(u) p_x(u) du$, Where μ_A is the membership function of A, p_x is the underlying (hidden) probability density of X and u is the generic value of X. In effect, a Z-valuation(X,A,B) may be viewed as a restriction on X defined by : $FEP(X \text{ is } A) \text{ is } B$ and more explicitly, possibility($FEP(X \in A) = u$) = $\mu_B(u)$.

Definition 4: Z-number

Zadeh defined a Z-number[1] associated with an uncertain variable X as an ordered pair (A, B) of two fuzzy numbers . A Z-number can be used to represent the information about an uncertain variable X of the type where the first number, A, is a restriction (constraint) on the values which a real-valued variable, X, can take and the second number, B, is a restriction on the degree of certainty that X is A (ie) measure of reliability (certainty) of the first component. Typically, A and B are described in a natural language. For example, X=travel time by car from Chennai to Madurai; A= approximately 10 hour; B=usually.

Definition 5: Z-valuation

Zadeh refers Z-valuation as an ordered triple, (X, A, B) and is equal to the assignment statement X is (A,B) which indicating that X takes the value A with probability equal B[1] .

Z-valuation provides us some information about the value of the associated variables. For example, travel time by car from Madurai to Thirumangalam, about 30 minutes, usually.

Z-valuation (X,A,B) may be viewed as a restriction on X([4],[5]) which is defined by

$FEP(X \text{ is } A) \text{ is } B$

More explicitly ,

$$\text{Possibility}(\text{FEP}(x \in A) = u) = \mu_B(u)$$

Equivalently, a Z-valuation (X, A, B) is a Z-restriction on X.

$$(X, A, B) \rightarrow X \text{ is } (A, B)$$

Definition 6: Z-restriction

A Z-restriction is expressed as $([4], [5])$

$R(X) : X \text{ is } Z$;

where Z is a combination of possibilistic and probabilistic restrictions defined as

$Z : \text{Prob}(X \text{ is } A) \text{ is } B$ (or) $\text{FEP}(X \text{ is } A) \text{ is } B$

in which A and B are fuzzy numbers.

Definition 7: Extension principle

Let f be a function from $U_1 \times U_2 \times \dots \times U_n$ to V. Let $A_1, A_2, \dots, A_n$ be fuzzy sets on $U_1, U_2, \dots, U_n$ respectively. Then extension principle indicates a method of associating a fuzzy set B on V based on the given information or inputs. This fuzzy set B is $B(v) = 0$, if $f^{-1}(v)$ is empty, and $B(v) = \max[\min\{A_1(u_1), A_2(u_2), \dots, A_n(u_n)\}]$, if $f^{-1}(v)$ is not empty, where the max is taken over all n-tuples such that $f(u_1, u_2, \dots, u_n) = v$. [8]

III. Combining Z-valuations

Yager had given the procedure for combining Z-valuations as follows:[2] Assume we have n Z-valuations on the variable X each on the form X is (A_i, B_i) for $i = 1$ to n. Each of these induces a possibility distribution G_i over the space of all probability distributions P. For $p \in P$ we have $G_i(p) = B(\int_{-\infty}^{\infty} A_i(x).p(x)dx)$. Now the overall information about the underlying probability distribution is obtained by combining these n possible distributions into a fused possibility distribution G on P. Various methods can be used, but the choice depends on the relationship between the pieces of information, the multiple Z-valuations

. One of these methods is to combine multiple Z-valuation perception of the value of X.

Assume Z-valuations: $X(A_1, B_1)$ and $X(A_2, B_2)$, each of these generates a possibility distribution G_i over the space P, $G_i(p) = B(\int_{-\infty}^{\infty} A_i(x).p(x)dx)$. Assume we assign a reliability of w_j to each of these and require that $w_1 + w_2 = 1$. Then the combined possibility distribution $G = w_1 G_1 \oplus w_2 G_2$. Using the extension principle and the rules of fuzzy arithmetic we then calculate $G(p)$ such that $G(p) = \max_{p=w_1 p_1 \oplus w_2 p_2} [G_1(p_1) \wedge G_2(p_2)]$ where $p = w_1 p_1 \oplus w_2 p_2$ is defined such that $p(x) = w_1 p_1(x) + w_2 p_2(x)$.

Another approach to combine Z-Valuations is to take the conjunction of the possibility distributions, here we get $G(p) = \min_j [G_j(p)]$. In this situation we are indicating that we must simultaneously accommodate all of the sources of information.

IV. Computation

We have two Z-valuations on one independent variable X as X is (A_1, B_1) and X is (A_2, B_2) . We are going to combine these two Z-valuations as shown below:

Example 1: Let us consider the following problem: It is known that the variable X has a underlying possibility distribution which is uniformly distribution with mean 50; the probability density function depends on parameter a. It is defines as follows:

$$P_a(x) = \begin{cases} \frac{1}{2a}, & [50 - a, 50 + a] \\ 0, & elsewhere \end{cases}$$

The parameter is known to be in the range [10, 30]. Assume we have two Z-valuations on the variable X of the form X is (A_1, B_1) and Y is (A_2, B_2) where A_1 and A_2 are the membership functions of X given by

$$\begin{aligned} A_1(x) &= 1 & \text{if } 20 \leq x \leq 30 & & A_2(x) &= \frac{1}{10}(x - 50) & \text{if } 50 \leq x \leq 60 \\ &= 1 - \frac{(x-30)}{10} & \text{if } 30 \leq x \leq 40 & & &= 1 & \text{if } 60 \leq x \leq 70 \end{aligned}$$

The possibility distributions B_1 and B_2 are given by

$$\begin{aligned} B_1(s) &= 20(s - .05) & \text{if } .05 \leq s \leq .1 & & B_2(t) &= 20(t - .3) & \text{if } .3 \leq t \leq .35 \\ &= 1 & \text{if } .1 \leq s \leq .15 & & &= 1 & \text{if } .35 \leq t \leq .4 \\ &= 1 - 20(s - .15) & \text{if } .15 \leq s \leq .2 & & &= 1 - 20(t - .4) & \text{if } .4 \leq t \leq .45 \end{aligned}$$

By first method:

Let $s = \text{FEP}(X \in A_1) = \int A_1(x) p_a(x) dx$

After calculation we get, for

$$\text{Case(i) } 10 \leq a \leq 20, s = \frac{(a-10)^2}{40a}$$

$$\text{Case (ii) } 20 \leq a \leq 30, s = \frac{1}{2} - \frac{15}{2a}$$

∴ we have $G(p_a) = \min(G_1(p_a), G_2(p_a))$

Case (i) $20 \leq a \leq 30$

$$s = \frac{1}{2} - \frac{15}{2a} \Rightarrow a = \frac{15}{2(.5-s)},$$

$$\therefore 20 \leq a \leq 30 \Leftrightarrow .125 \leq s \leq .25$$

$$\text{For } 20 \leq a \leq 21.43 \Leftrightarrow .125 \leq s \leq .15, \text{ we have } G_1(p_a) = B_1(s) = 1$$

$$\text{For } 21.43 \leq a \leq 25 \Leftrightarrow .15 \leq s \leq .2, \text{ we have } G_1(p_a) = B_1(s) = 1 - 20(s - .15) = \frac{150}{a} - 6$$

$$\text{For } 25 \leq a \leq 30 \Leftrightarrow .2 \leq s \leq .25, \text{ we have } G_1(p_a) = B_1(s) = 0$$

Case (ii) $10 \leq a \leq 20$

$$s = \frac{(a-10)^2}{40a}$$

$$\therefore 10 \leq a \leq 20 \Leftrightarrow 0 \leq s \leq .125$$

$$\text{For } 10 \leq a \leq 15.58 \Leftrightarrow 0 \leq s \leq .05, \text{ we have } G_1(p_a) = B_1(s) = 0$$

$$\text{For } 15.58 \leq a \leq 18.63 \Leftrightarrow .05 \leq s \leq .1, \text{ we have } G_1(p_a) = B_1(s) = 20(s - .05) = \frac{(a-10)^2}{2a} - 1$$

$$\text{For } 18.63 \leq a \leq 20 \Leftrightarrow 0.1 \leq s \leq .125, \text{ we have } G_1(p_a) = B_1(s) = 1$$

Case (i) If $10 \leq b \leq 20$, $t = \frac{b-5}{2b}$

$$\therefore 10 \leq b \leq 20 \Leftrightarrow 0.25 \leq t \leq .375$$

$$\text{For } 10 \leq b \leq 12.5 \Leftrightarrow 0.25 \leq t \leq .3, \text{ we have } G_1(p_a) = B_1(s) = 0$$

$$\text{For } 12.5 \leq b \leq 16.67 \Leftrightarrow .3 \leq t \leq .35, \text{ we have } G_1(p_a) = B_1(s) = 20(t - .3) = 4 - \frac{50}{b}$$

$$\text{For } 16.67 \leq b \leq 20 \Leftrightarrow .35 \leq t \leq .375, \text{ we have } G_1(p_a) = B_1(s) = 1$$

Case (ii) If $20 \leq b \leq 30$, $t = \frac{15}{2b}$

$$\therefore 20 \leq b \leq 30 \Leftrightarrow 0.25 \leq t \leq .375$$

$$\text{For } 20 \leq b \leq 21.43 \Leftrightarrow 0.35 \leq t \leq .375, \text{ we have } G_1(p_b) = B_1(s) = 1$$

$$\text{For } 21.43 \leq b \leq 25 \Leftrightarrow 0.3 \leq t \leq .35, \text{ we have } G_1(p_b) = B_1(s) = \frac{150}{b} - 6$$

$$\text{For } 25 \leq b \leq 30 \Leftrightarrow t \leq .3, \text{ we have } G_1(p_b) = B_1(s) = 0$$

So we have,

	$G_1(p_a)$		$G_2(p_a)$
$10 \leq a \leq 15.58$	0	$10 \leq a \leq 12.5$	0
$15.58 \leq a \leq 18.63$	$\frac{(a-10)^2}{2a} - 1$		
$18.63 \leq a \leq 20$	1	$12.5 \leq a \leq 16.67$	$4 - \frac{50}{a}$
		$16.67 \leq a \leq 20$	1
$20 \leq a \leq 21.43$	1	$20 \leq a \leq 21.43$	1
$21.43 \leq a \leq 25$	$\frac{150}{a} - 6$	$21.43 \leq a \leq 25$	$\frac{150}{a} - 6$
$25 \leq a \leq 30$	0	$25 \leq a \leq 30$	0

$$\text{In } 10 \leq a \leq 12.5, G(p_a) = \min(G_1(p_a), G_2(p_a)) = \min(0, 0) = 0$$

$$\text{In } 12.5 < a \leq 18.63, G(p_a) = \min(\frac{(a-10)^2}{2a} - 1, 4 - \frac{50}{a}) = 4 - \frac{50}{a}$$

$$\text{In } 18.63 < a \leq 20, G(p_a) = \min(1, 1) = 1$$

$$\text{In } 20 < a \leq 21.43, G(p_a) = \min(1, 1) = 1$$

$$\text{In } 21.43 < a \leq 25, G(p_a) = \min(\frac{150}{a} - 6, \frac{150}{a} - 6) = \frac{150}{a} - 6$$

$$\text{In } 25 < a \leq 30, G(p_a) = \min(0, 0) = 0$$

Intuitively the second method is more appealing. The second method shows that combining the two Z-valuations leads to a narrower range for the parameter a. The range is a fuzzy interval whose core is [18.63, 21.43] and support is [12.5, 25].

V. Approximation using alpha cuts

Let us consider the case where the underlying probability density function is dependent on a single parameter say 'a'. Alternate way to combine Z-Valuations is by computation of Approximate Z-Valuations using α -cuts. Assume

Z-valuation: $X(A_1, B_1)$ and $X(A_2, B_2)$. To find approximate Z-valuations, we proceed as follows: As we already seen, $B_1(s)$ and $B_2(t)$ generates possibility distribution G_i over the space of probability distribution P as $G_1(p_a) = B_1(\int A_1(x)p_a(x)dx)$ and $G_2(p_a) = B_2(\int A_2(x)p_a(x)dx)$ where $p(x)$ is the probability distribution of X . Suppose the α -cuts of B_1, B_2 are $[S_0, S_1], [T_0, T_1]$ respectively. Then $B_1(s) \geq \alpha$ if $s \in [S_0, S_1]$ and $B_2(s) \geq \alpha$ if $t \in [T_0, T_1]$. Then if

$$R_1(\alpha) = \{a \mid S_0 \leq \int A_1(x)p_a(x)dx \leq S_1\}$$

$$R_2(\alpha) = \{a \mid T_0 \leq \int A_2(x)p_a(x)dx \leq T_1\}$$

then $R_1(\alpha) \cap R_2(\alpha)$ is the alpha cut for the combined distribution $G(p_a)$. The α -cut of $G(p_a)$ corresponding to various values of α , say $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$ can be determined. This gives the approximation to $G(p_a)$.

Example 2:

Let us consider,

X be the normal random variate with μ and σ as parameters.

Normal distribution $N(\mu, \sigma^2)$ with $\sigma = 10$

Membership functions are

$$A_1(x) = \frac{x-370}{10}, \quad 370 \leq x \leq 380$$

$$= 1, \quad 380 \leq x \leq 390$$

$$= 1 - \frac{x-390}{10}, \quad 390 \leq x \leq 400$$

$$A_2(x) = 1, \quad 400 \leq x \leq 420$$

$$= 1 - \frac{x-420}{20}, \quad 420 \leq x \leq 440$$

The possibility distributions of s and t are,

$$B_1(s) = \begin{cases} 40(s - .25), & 0.25 \leq s \leq 0.275 \\ 1, & 0.275 \leq s \leq 0.325 \\ 1 - 40(s - 0.325), & 0.325 \leq s \leq 0.35 \end{cases}$$

$$B_2(t) = \begin{cases} 40(t - .45), & 0.45 \leq t \leq 0.475 \\ 1, & 0.475 \leq t \leq 0.525 \\ 1 - 40(t - 0.525), & 0.525 \leq t \leq 0.55 \end{cases}$$

$$\text{Let } s = \text{FEP}(X \in A) = \int_{370}^{400} \mu_A(x) f(x) dx$$

$$\text{After calculation we get, } s = \frac{\sigma}{10\sqrt{2\pi}} \left(e^{-\frac{Z_1^2}{2}} - e^{-\frac{Z_2^2}{2}} \right) + \left(\frac{\mu-370}{10} \right) \text{prob}(Z_1 \leq z \leq Z_2) +$$

$$\text{prob}(Z_2 \leq z \leq Z_3) + \left(\frac{400-\mu}{10} \right) \text{prob}(Z_3 \leq z \leq Z_4) + \frac{\sigma}{10\sqrt{2\pi}} \left(e^{-\frac{Z_4^2}{2}} - e^{-\frac{Z_3^2}{2}} \right)$$

$$\text{Where } Z_1 = \frac{370-\mu}{\sigma}; Z_2 = \frac{380-\mu}{\sigma}; Z_3 = \frac{390-\mu}{\sigma}; Z_4 = \frac{400-\mu}{\sigma}$$

$$\text{Let } t = \text{FEP}(X \in B) = \int_{400}^{440} \mu_B(x) f(x) dx$$

After calculation we get,

$$t = \text{prob}(Z_5 \leq z \leq Z_6) + \left(\frac{440-\mu}{20} \right) \text{prob}(Z_6 \leq z \leq Z_7) + \frac{\sigma}{20\sqrt{2\pi}} \left(e^{-\frac{Z_7^2}{2}} - e^{-\frac{Z_6^2}{2}} \right)$$

$$\text{Where } Z_5 = \frac{400-\mu}{\sigma}; Z_6 = \frac{420-\mu}{\sigma}; Z_7 = \frac{440-\mu}{\sigma}$$

Calculating s and t for different values of μ and tabulating $B_1(s)$ and $B_2(t)$ (Figure (i) and Figure (ii)) corresponding to various values of s and t .

Figure(i) Shows $B_1(s)$

μ	s	$B_1(s)$
360	0.0743	0
365	0.1666	0
368	0.2461	0
368.25	0.2521	0.084
368.4	0.258	0.32
368.5	0.2611	0.444
368.6	0.2645	0.58
368.8	0.2695	0.78
368.9	0.2741	0.964
369	0.2763	1
369.5	0.2919	1

369.7	0.298	1
369.8	0.3011	1
370	0.3078	1
370.5	0.3236	1
370.6	0.327	0.92
370.7	0.3304	0.784
370.8	0.3333	0.668
370.9	0.3368	0.528
371	0.3399	0.404
371.1	0.3434	0.264
371.3	0.3502	0
380	0.6090	0
390	0.6095	0
398	0.3730	0
398.7	0.3502	0
398.9	0.3434	0.264
399	0.3399	0.404
399.1	0.3368	0.528
399.2	0.3333	0.668
399.3	0.3304	0.784
399.4	0.327	0.92
399.5	0.3236	1
400	0.3078	1
400.2	0.3011	1
400.3	0.298	1
400.5	0.2919	1
401	0.2763	1
401.1	0.2741	0.964
401.2	0.2695	0.78
401.4	0.2645	0.58
401.5	0.2611	0.444
401.6	0.258	0.32
401.75	0.2521	0.084
402	0.2461	0
405	0.1666	0
410	0.0743	0

Figure(ii) Shows $B_2(t)$

μ	t	$B_2(t)$
398.5	0.438	0
398.9	0.4531	0.062
399	0.457	0.140
399.1	0.4608	0.432
399.2	0.4648	0.592
399.3	0.4686	0.744
399.4	0.4725	0.9
399.5	0.477	1
400	0.4959	1
400.1	0.4997	1
400.2	0.5037	1
400.5	0.515	1
400.7	0.5228	1
400.9	0.5306	0.776
401	0.5344	0.624
401.1	0.5386	0.456
401.2	0.5421	0.316
401.4	0.5496	0.016
401.5	0.5535	0
401.6	0.5573	0
410	0.7977	0
420	0.7819	0
428	0.5653	0
428.5	0.549	0.04
428.6	0.545	0.2
428.7	0.5422	0.312
428.8	0.5389	0.444
428.9	0.535	0.6
429	0.5322	0.712
429.2	0.5255	0.98
429.3	0.5222	1
429.5	0.5155	1

429.8	0.505	1
430	0.4987	1
430.5	0.4819	1
430.7	0.475	1
430.8	0.4717	0.868
430.9	0.4683	0.732
431	0.4648	0.592
431.1	0.4622	0.488
431.2	0.4583	0.332
431.3	0.4548	0.192
431.5	0.4481	0
440	0.1953	0

From the above table, we are able to find the α –cuts for possibility distribution of s, $G_1(s)$ and the corresponding $R_1(\alpha)$ as follows:

$$\begin{aligned} R_1(1) &= [369, 370.5] \cup [399.5, 401] \\ R_1(0.964) &= [368.9, 370.5] \cup [399.5, 401.1] \\ R_1(0.78) &= [368.8, 370.7] \cup [399.3, 401.2] \\ R_1(0.528) &= [368.6, 370.9] \cup [399.1, 401.4] \\ R_1(0.404) &= [368.5, 371] \cup [399, 401.5] \\ R_1(0.264) &= [368.4, 371.1] \cup [398.9, 401.6] \\ R_1(0.084) &= [368.25, 371.1] \cup [398.9, 401.75] \\ R_1(0.02) &= [368.25, 371.1] \cup [398.9, 401.75] \end{aligned}$$

From the above table, we are able to find the α - cuts for possibility distribution of t, $G_2(t)$ and the corresponding $R_2(\alpha)$ as follows:

$$\begin{aligned} R_2(1) &= [399.5, 400.7] \cup [429.3, 430.7] \\ R_2(0.964) &= [399.5, 400.7] \cup [429.2, 430.7] \\ R_2(0.78) &= [399.4, 400.9] \cup [429.2, 430.8] \\ R_2(0.528) &= [399.2, 401] \cup [428.9, 431] \\ R_2(0.404) &= [399.1, 401.1] \cup [428.8, 431.1] \\ R_2(0.264) &= [399.1, 401.2] \cup [428.7, 431.2] \\ R_2(0.084) &= [399, 401.2] \cup [428.6, 431.3] \\ R_2(0.02) &= [398.9, 401.4] \cup [428.5, 431.3] \end{aligned}$$

The α - cut of the possibility distribution $G(\mu)$ is $R_1(\alpha) \cap R_2(\alpha)$.

For $\alpha = 1$, the α - cut is $[399.5, 400.7]$
 For $\alpha = 0.964$, the α - cut is $[399.5, 400.7]$
 For $\alpha = 0.78$, the α - cut is $[399.4, 400.9]$
 For $\alpha = 0.528$, the α - cut is $[399.2, 401]$
 For $\alpha = 0.404$, the α - cut is $[399.1, 401.1]$
 For $\alpha = 0.264$, the α - cut is $[399.1, 401.2]$
 For $\alpha = 0.084$, the α - cut is $[399, 401.2]$
 For $\alpha = 0.02$, the α - cut is $[398.9, 401.4]$

Hence it is clear that the value of the parameter μ lies in a fuzzy interval whose core is $[399.5, 400.7]$ and support is $[398.7, 401.5]$.

VI. Conclusion

When we have two reliable z-valuations, the method of taking conjunction of the possibility distributions seems to be the better method; it is practical and leads to useful results.

In case exact methods cannot be employed then the approximation method using alpha cuts which has been described in the previous section can be used.

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