



SOME NORMAL SOLUTION ON OPERATOR EQUATION

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Abstract:

An operator means a bounded linear operator on a Hilbert Space. This paper proves the assertion made in its title. Following theorem yields the famous result

$$AB + BA^* = I = A^*B + BA \quad (1.1)$$

Where A and B are bounded linear operator on a Hilbert Space H.

THEOREM (1.1): – If $\sigma(R_e B) \cap \sigma(-R_e B) = \emptyset$ then the solution A to (1.1) is normal.

THEOREM (1.2): – The solution A to (1.1) is normal if any one of the following conditions is satisfied.

$$(i) \quad 0 \notin W(R_e, B)$$

$$(ii) \quad 0 \notin W(I MB)$$

THEOREM (1.3): – If $AB + BA^* = I$ has a solution B, then $0 \notin W\sigma_s(A)$ and $0 \in P(R_e B)$

$$\text{Further more } \|(R_e B)^{-1}\| \leq \|A\|$$

THEOREM (1.4): – If $\sigma(A) \subset X_+$ then there exist a positive definite solution B to (1.1) such that

$$(i) \quad B = D U \text{ for some positive definite operator D and U.}$$

$$(ii) \quad [U, A] = 0$$

Assuming the existence of solution B to (1.1) it has been shown that the homogeneous forms

$$AX + XA^* = 0, \quad A^*X + XA = 0 \quad (1.2)$$

of equation (1.1) play important role in finding unique solution.

They have also proved the following results.

Lemma :- (1.1) If $0 \notin W(R_e A)$ then $X = 0$ is the only solution of equation (1.2).

Lemma :- (1.2) If $X = 0$ is the unique solution of equation (1.2) then the solution B of (1.1) is unique.

THEOREM (1.4): If $0 \notin W(R_e A)$, then the solution B of (1.1) is unique more over $B^* = B$.

Now we give the following important results, in the proof of our theorem.

THEOREM (1.5): Let B be a normal operator and A be an operator such that $0 \notin W(A)$. If $[B, A^*A] = 0$ and $[B, A^*BA] = 0$ then $[B, A] = 0$

THEOREM (1.6): If A and B are the bounded operators on Hilbert Space H with B self-adjoint satisfying the operator equation (1.1) the following result hold.

$$(i) \quad [B^2, A] = 0$$

$$(ii) \quad [B^2, A^*] = 0$$

$$(iii) \quad [B^2, A^* + A] = 0$$

$$(iv) \quad [B, A^* - A] = 0$$

$$(v) \quad BA^*A = AA^*B$$

$$(vi) \quad BA A^* = A^*AB$$

Proof: (1) From equation (1.1) we have $B^2A + BA^*B = B$ i.e. $B^2A = (I - BA^*)B$

$$\text{i.e. } B^2A = (AB)B = AB^2$$

$$\text{i.e. } B^2A - AB^2 = 0$$

$$\Rightarrow [B^2, A] = 0$$

$$(ii) \quad \text{We have}$$

(1.3)

$$\begin{aligned}
 & BAB + B^2 A^* = B \text{ i.e. } B^2 A^* = (I - AB)B \\
 \text{i.e. } & B^2 A^* = (A^* B)B = A^* B^2 \\
 \text{i.e. } & B^2 A^* - A^* B^2 = 0 \\
 \Rightarrow & [B^2, A^*] = 0 \\
 \text{(iii)} & \text{ Now from the equation (1.3) and (1.4) we have} \\
 & B^2(A^* + A) - (A^* + A)B^2 = 0 \\
 \Rightarrow & [B^2, A^* + A] = 0 \\
 \text{(iv)} & \text{ We have} \\
 & [B, A^* - A] = B(A^* - A) - (A^* - A)B \\
 & = BA^* - BA - A^*B + AB \\
 & = I - AB - BA - (I - BA) + AB
 \end{aligned} \tag{1.4}$$

$$\begin{aligned}
 \text{From equation (1.1)} & \\
 & = I - AB - BA - I + BA + AB \\
 & = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} & \text{ From equation (1.1) we have} \\
 & BA^* A = A - ABA = A(I - BA) = A A^* B \\
 \text{i.e. } & BA^* A = AA^* B
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} & \text{ We have} \\
 & BAA^* = A^* - A^* BA^* = A^*(I - BA^*) = A^* AB \\
 \Rightarrow & BAA^* = A^* AB.
 \end{aligned}$$

Following results have been obtained after replacing $(R_e B)$ by B in the condition $\sigma(R_e B) \cap \sigma(-R_e B) = \phi$ of those given in

Theorem (1.1) and also in theorem (1.2)

THEOREM (1.7): If $\sigma(B) \cap \sigma(-B) = \phi$ then the solution A to equation (1.1) is normal.

Proof: From the result (v) and (vi) of the (1.6). we have

$$BA^* A = A A^* B \text{ and } BA A^* = A^* AB.$$

$$\text{Let } A^* A - AA^* = X$$

$$\begin{aligned}
 \text{Then } BX + XB &= B(A^* A - AA^*) + (A^* A - AA^*)B \\
 &= (B A^* A - AA^* B) - (BAA^* - A^* AB) \\
 &= 0
 \end{aligned}$$

Clearly $X = 0$ is solution of the equation $BX + XB = 0$. Since $\sigma(B) \cap \sigma(-B) = \phi$, hence by [(Rosenblum 1956). If $\sigma(C) \cap \sigma(-C) = \phi$ then the equation $CX + XC = 0$ has $X = 0$ the only solution] $X = 0$ is the only solution. Hence $A^* A - AA^* = 0$; i.e. $A^* A = AA^*$; i.e. A is normal.

THEOREM (1.8): The solution to equation (1.1) is normal if $0 \notin W(B)$.

Proof: Let

$$T = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & A^* A \\ AA^* & 0 \end{bmatrix}$$

Let $X = x_1 \oplus x_2$ where $x_1, x_2 \in H$ such that $\|x\| = 1$. We suppose that $0 \notin W(B)$; i.e. $(Bx, x) \neq 0$ for all $x \in H$ we have

$$\begin{aligned}
 (Tx, x) &= \left(\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} (x_1 \oplus x_2), (x_1 \oplus x_2) \right) \\
 \text{or, } (Tx, x) &= ((Bx_1 + 0) \oplus (0 + Bx_2), (x_1 \oplus x_2)) \\
 &= ((Bx_1 + Bx_2) x_1 \oplus x_2) \\
 &= (B(x_1 \oplus x_2), (x_1 \oplus x_2)) \\
 &= (Bx, x) \neq 0 \\
 \Rightarrow (Tx, x) &\neq 0 \quad \text{i.e.} \quad 0 \notin W(T)
 \end{aligned}$$

Also we have

$$TK = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & A^*A \\ AA^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & BA^*A \\ BA^*A & 0 \end{bmatrix}$$

$$K^*T = \begin{bmatrix} 0 & AA^* \\ A^*A & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & AA^*B \\ A^*AB & 0 \end{bmatrix}$$

By (v) and (vi) we have $TK = K^*T$. Similarly we have $TK^* = KT$. The operator $(K - K^*)$ is normal and it is easily verify that $[T^*T, K - K^*] = 0$ and $[T^*(K - K^*)T, K - K^*] = 0$. Hence by theorem [(Nakamoto, 1977) Let B be a normal operator and A be an operator such that $0 \notin W(A)$. If $[B, A^*A] = 0$ and $[B, A^*BA] = 0$ then $[B, A] = 0$] of it follows that $[T, K - K^*] = 0$ i.e. $TK - TK^* = KT - K^*T$ i.e. $TK - TK^* = TK^* - TK$ i.e. $TK - TK^* = 0$ i.e. $T(K - K^*) = 0$, Since $0 \notin W(T)$. This implies that $K - K^* = 0$, i.e. $K = K^*$, i.e. $A^*A = AA^*$. Hence A is normal.

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