



Analysis of Successive Occurrence of Digit 1 in Natural Numbers Less Than 10^n

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Abstract: All positive integers less than 10^n are under consideration. 1 is the first natural number as well as non-zero digit symbol. A detail analysis of successive occurrence of digit 1 in numbers less than 10^n is done in this work. The formula for the number of successive occurrences of 1's is determined. The first instance of successive 1's is easily predictable. The last occurrence is also formulated. All the analysis is extended ahead to multiple number of successive occurrences of 1's. Finally, all results are generalized for successive occurrences of all non-zero digits.

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I. Introduction

Positive integers form infinite list

1, 2, 3, ...

which are studied in branch of Mathematics called Number Theory. They are so basic that their applications are spread in every branch of Mathematics as well as other subjects.

Beginning nine members 1, 2, 3, 4, 5, 6, 7, 8, 9 are digits as well. One more digit in place value system is 0.

In the present work, the word number is used to mean natural number. We use the new convention that 0 itself is not in the set of Natural numbers N and consider the ranges $1 - 10^n$, except 10^n , for $n \in N$. The numbers under consideration are m , with $1 \leq m < 10^n$. The last number 10^n is not taken as it begins next higher range of number of digits $(n + 1)$.

II. Successive Occurrence of Digit 1

The first natural number is the digit 1. It's one more property is that it is there in number systems with all bases [1]. Detail analysis of occurrences of digit 1, in fact, every digit d with $1 \leq d \leq 9$, is given in [2]. Here, the successive occurrence of digit 1 is analyzed in the range of $1 - 10^n$, except the last number 10^n , for all natural numbers n .

We have determined counts of successive occurrence of single 1 and 2 1's in numbers just one less than one quintillion, i.e., 10^{18} , by using a computer language Java program and they are as given below.

Table I : Number of Numbers with Single and Double Successive 1's in their Digits

Sr. No.	Numbers Range Less Than	Number of Numbers with Single (Successive) 1	Number of Numbers with two Successive 1's
1.	10^1	1	0
2.	10^2	18	1
3.	10^3	243	18
4.	10^4	2,916	243
5.	10^5	32,805	2,916
6.	10^6	354,294	32,805
7.	10^7	3,720,087	354,294
8.	10^8	38,263,752	3,720,087
9.	10^9	387,420,489	38,263,752
10.	10^{10}	3,874,204,890	387,420,489
11.	10^{11}	38,354,628,411	3,874,204,890
12.	10^{12}	376,572,715,308	38,354,628,411
13.	10^{13}	3,671,583,974,253	376,572,715,308

Sr. No.	Numbers Range Less Than	Number of Numbers with Single (Successive) 1	Number of Numbers with two Successive 1's
14.	10^{14}	35,586,121,596,606	3,671,583,974,253
15.	10^{15}	343,151,886,824,415	35,586,121,596,606
16.	10^{16}	3,294,258,113,514,384	343,151,886,824,415
17.	10^{17}	31,501,343,210,481,297	3,294,258,113,514,384
18.	10^{18}	300,189,270,593,998,260	31,501,343,210,481,297

In the first range $1 \leq m < 10^1 = 10$, single 1 occurs just once itself as a number. It's count can be seen as ${}^{1-(1-1)}C_1 9^{1-1} = 1$. Single occurrence is qualified to be treated as successive by absence of non-successive character.

In the second range $1 \leq m < 10^2 = 100$, single 1 comes 18 times. Its 9 occurrences are in numbers

1, 21, 31, 41, 51, 61, 71, 81, and 91,

at unit's places and 9 are in numbers

10, 12, 13, 14, 15, 16, 17, 18, and 19,

at ten's places. So, second block has it ${}^{2-(1-1)}C_1 9^{2-1} = 2 \times 9 = 18$ times. As stated earlier, it is considered to be successive.

In this range, double 1 comes only once in number 11. Clearly this is of successive type. It's count can be seen as ${}^{2-(2-1)}C_1 9^{2-2} = 1 \times 1 = 1$ time.

In the third range, $1 \leq m < 10^3 = 1,000$, single 1 comes 243 times in numbers

1, 21, 31, ..., 91, 201, 221, ..., 291, 301, 321, ..., 391, ..., 901, 921, ..., 991,

at unit's places

12, 13, 14, ..., 19, 210, 212, ..., 219, 310, 312, ..., 319, ..., 910, 912, ..., 919,

at ten's places and

100, 102, 103, ..., 109, 120, 122, 123, ..., 130, 132, 133, ..., 139, ..., 190, 192, 193, ..., 199

at hundred's places.

This occurrence count in third block is ${}^{3-(1-1)}C_1 9^{3-1} = 3 \times 9^2 = 3 \times 81 = 243$, all of them being considered successive.

In this range, successive double 1's occur in

211, 311, ..., 911,

at unit's and ten's places and in

110, 112, 113, ..., 119

at ten's and hundred's places.

Their count is ${}^{3-(2-1)}C_1 9^{3-2} = 2 \times 9 = 18$ times.

Also successive triple 1's occur once in 111.

We can explain all numbers in above table this way.

There is similar explanation for successive occurrences of multiple 1's in these ranges. There is clearly a systematic pattern in the resulting figures. We have formulated it by introducing a notation first.

Notation : We introduce the notation ${}_1^S O_r^n$ for number of numbers less than 10^n containing r number of successive 1's.

Theorem 1 : If r and n are positive integers with $r \leq n$, then the number of numbers containing exactly r number of successive digit 1's in the range $1 \leq m < 10^n$ is

$${}_1^S O_r^n = {}^{n-(r-1)}C_1 9^{n-r}.$$

Proof. Let n and r be positive integers with $r \leq n$. There are in total 10 digits, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, available to occupy n places in all numbers in range $1 \leq m < 10^n$. We want successive r places to be occupied by digit 1. The various choices for these successive r places for digit 1 will be ${}^{n-(r-1)}C_1$ in number. Now for each such choice, remaining $n - r$ places are to be occupied by any of the remaining 9 digits except 1 and there are 9^{n-r} choices for each of that. This totals to ${}^{n-(r-1)}C_1 9^{n-r}$ and hence ${}_1^S O_r^n = {}^{n-(r-1)}C_1 9^{n-r}$. This completes the proof of the theorem.

The table given above can be extended to higher occurrences of successive 1's by using this formula.

Table II : Number of Numbers with Multiple Successive 1's in their Digits

Sr. No.	Numbers Range <	Number of Numbers with 3 Successive 1's	Number of Numbers with 4 Successive 1's	Number of Numbers with 5 Successive 1's
1.	10^3	1	0	0
2.	10^4	18	1	0
3.	10^5	243	18	1
4.	10^6	2,916	243	18
5.	10^7	32,805	2,916	243
6.	10^8	354,294	32,805	2,916

Sr. No.	Numbers Range <	Number of Numbers with 3 Successive 1's	Number of Numbers with 4 Successive 1's	Number of Numbers with 5 Successive 1's
7.	10^9	3,720,087	354,294	32,805
8.	10^{10}	38,263,752	3,720,087	354,294
9.	10^{11}	387,420,489	38,263,752	3,720,087
10.	10^{12}	3,874,204,890	387,420,489	38,263,752
11.	10^{13}	38,354,628,411	3,874,204,890	387,420,489
12.	10^{14}	376,572,715,308	38,354,628,411	3,874,204,890
13.	10^{15}	3,671,583,974,253	376,572,715,308	38,354,628,411
14.	10^{16}	35,586,121,596,606	3,671,583,974,253	376,572,715,308
15.	10^{17}	343,151,886,824,415	35,586,121,596,606	3,671,583,974,253
16.	10^{18}	3,294,258,113,514,384	343,151,886,824,415	35,586,121,596,606

Sr. No.	Numbers Range <	Number of Numbers with 6 Successive 1's	Number of Numbers with 7 Successive 1's	Number of Numbers with 8 Successive 1's	Number of Numbers with 9 Successive 1's
1.	10^6	1	0	0	0
2.	10^7	18	1	0	0
3.	10^8	243	18	1	0
4.	10^9	2,916	243	18	1
5.	10^{10}	32,805	2,916	243	18
6.	10^{11}	354,294	32,805	2,916	243
7.	10^{12}	3,720,087	354,294	32,805	2,916
8.	10^{13}	38,263,752	3,720,087	354,294	32,805
9.	10^{14}	387,420,489	38,263,752	3,720,087	354,294
10.	10^{15}	3,874,204,890	387,420,489	38,263,752	3,720,087
11.	10^{16}	38,354,628,411	3,874,204,890	387,420,489	38,263,752
12.	10^{17}	376,572,715,308	38,354,628,411	3,874,204,890	387,420,489
13.	10^{18}	3,671,583,974,253	376,572,715,308	38,354,628,411	3,874,204,890

Sr. No.	Numbers Range <	Number of Numbers with 10 Successive 1's	Number of Numbers with 11 Successive 1's	Number of Numbers with 12 Successive 1's	Number of Numbers with 13 Successive 1's
1.	10^9	0	0	0	0
2.	10^{10}	1	0	0	0
3.	10^{11}	18	1	0	0
4.	10^{12}	243	18	1	0
5.	10^{13}	2,916	243	18	1
6.	10^{14}	32,805	2,916	243	18
7.	10^{15}	354,294	32,805	2,916	243
8.	10^{16}	3,720,087	354,294	32,805	2,916
9.	10^{17}	38,263,752	3,720,087	354,294	32,805
10.	10^{18}	387,420,489	38,263,752	3,720,087	354,294

Sr. No.	Numbers Range <	Number of Numbers with 14 Successive 1's	Number of Numbers with 15 Successive 1's	Number of Numbers with 16 Successive 1's	Number of Numbers with 17 Successive 1's	Number of Numbers with 18 Successive 1's
1.	10^{14}	1	0	0	0	0
2.	10^{15}	18	1	0	0	0
3.	10^{16}	243	18	1	0	0
4.	10^{17}	2,916	243	18	1	0
5.	10^{18}	32,805	2,916	243	18	1

The nature of the formula giving these numbers is such that the columns are gradually shifted down.

III. First Successive Occurrence of Digit 1

The first number containing 1 is obviously 1. Containing 2 successive 1's, the first number is 11, for 3 it is 111 and so on. We have formulated numbers with first successive 1's.

Formula 1 : If n and r are natural numbers, then the first occurrence of r number of successive 1's in numbers in range $1 \leq m < 10^n$ is

$$f = \begin{cases} - & , \text{ if } r > n \\ \sum_{j=0}^{r-1} (1 \times 10^j) & , \text{ if } r \leq n \end{cases}$$

IV. Last Successive Occurrence of Digit 1

The last successive occurrences of 1 in our ranges are also determined.

Table III : The Last Occurrences of Multiple Successive 1's in the Digits of Numbers

Sr. No.	Number Range $\leftarrow \rightarrow$ Last Number with Successive $\downarrow$	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
1.	1 1	1	91	991	9,991	99,991	999,991	9,999,991	99,999,991	999,999,991
2.	2 1's	-	11	911	9,911	99,911	999,911	9,999,911	99,999,911	999,999,911
3.	3 1's	-	-	111	9,111	99,111	999,111	9,999,111	99,999,111	999,999,111
4.	4 1's	-	-	-	1,111	91,111	991,111	9,991,111	99,991,111	999,991,111
5.	5 1's	-	-	-	-	11,111	911,111	9,911,111	99,911,111	999,911,111
6.	6 1's	-	-	-	-	-	111,111	9,111,111	99,111,111	999,111,111
7.	7 1's	-	-	-	-	-	-	1,111,111	91,111,111	991,111,111
8.	8 1's	-	-	-	-	-	-	-	11,111,111	911,111,111
9.	9 1's	-	-	-	-	-	-	-	-	111,111,111

We have formulated them.

Formula 2 : If n and r are natural numbers, then the last occurrence of r successive 1's in numbers in range $1 \leq m < 10^n$ is

$$l = \begin{cases} - & , \text{ if } r > n \\ 0 & , \text{ if } r = n \\ \sum_{j=0}^{r-1} (1 \times 10^j) + \sum_{j=r}^{n-1} (9 \times 10^j) & , \text{ if } r < n \end{cases}$$

Some integer sequences occurring here are analogous to those in [2].

V. Extension to Other Non-zero Digits

Finally, we conclude by mentioning an important thing that whatever discussion has been done for occurrences of successive digit 1's is parallelly applicable for other non-zero digits 2 through 9. Denoting the non-zero digit of interest by d , where $1 \leq d \leq 9$, again the range under consideration is $1 \leq m < 10^n$ and $1 \leq r \leq n$.

Notation : We generalize the notation ${}_d^S O_r^n$ for number of numbers less than 10^n with r number of successive digits d 's.

Theorem 2 : If r , n and d are positive integers with $r \leq n$ and $1 \leq d \leq 9$, then the number of numbers containing exactly r number of successive digit d 's in the range $1 \leq m < 10^n$ is

$${}_d^S O_r^n = {}^{n-(r-1)} C_1 9^{n-r}$$

Proof. The presence of each digit d , with $1 \leq d \leq 9$, is same for all numbers in the total range $1 \leq m < 10^n$, and hence the proof is same as that for Theorem 1.

Formula 3 : If r , n and d are positive integers with $1 \leq d \leq 9$, then the first occurrence of r number of successive d 's in numbers in range $1 \leq m < 10^n$ is in number

$$f = \begin{cases} - & , \text{ if } r > n \\ \sum_{j=0}^{r-1} (d \times 10^j) & , \text{ if } r \leq n \end{cases}$$

Formula 4 : If r , n and d are positive integers with $1 \leq d \leq 9$, then the last occurrence of r number of successive d 's in numbers in range $1 \leq m < 10^n$ is in number

$$l = \begin{cases} - & , \text{ if } r > n \\ 0 & , \text{ if } r = n \\ \sum_{j=0}^{r-1} (d \times 10^j) + \sum_{j=r}^{n-1} (9 \times 10^j) & , \text{ if } r < n \end{cases}$$

It is noteworthy that the all Formulae 1, 2, 3, and 4 provided here are exactly same as formulae with corresponding numbers in [2] as the first and last occurrences of any non-zero digit happen to be successive ones.

References

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