



A PEXIDER DIFFERENCE FOR A (S, N)-CONVEX PEXIDER FUNCTIONAL EQUATION

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Abstract: In this paper we investigate the Hyers-Ulam stability of the following functional equation $F_k(x_1, \dots, \alpha(a_k) + \beta(b_k), \dots, x_n) = \alpha^s H_k(x_1, \dots, x_k, \dots, x_n) + \beta^s H_k(x_1, \dots, b_k, \dots, x_n)$ in topological vector spaces, where F, H are functions defined on a groups with values in a topological vector spaces.

Keywords: Hyers-Ulam Stability, Additive Mapping, Quadratic Mapping, Topological Vector Space.

I. Introduction and Preliminaries

In 1940 S.Ulam [6] proposed the general Ulam stability problem: Let G_1 be a group, G_2 a metric group with the metric d . Given $\varepsilon > 0$, does there exists $\delta > 0$ such that if a function $h: G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy) - h(x)h(y)) < \delta, \quad (x, y \in G_1),$$

then there is a homomorphism $H: G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon, \quad (x \in G_1)?$$

D. H. Hyers [3] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, a generalized version of Hyers' theorem for approximate additive mappings was given by T.

Aoki [2]. In 1978, Th. M. Rassias [5] extended the theorem of Hyers by considering the unbounded cauchy difference inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (\varepsilon \geq 0, p \in [0,1)) \quad (1.1)$$

Recently, Adam and Czerwik [1] investigated the problem of the Hyers-Ulam stability of a generalized quadratic functional equation in linear topological spaces. Najati and Moghimi [4] investigated the Hyers-Ulam stability of the functional equation

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 2f(2x) - 2f(x)$$

in quasi-Banach spaces.

Let G be an abelian group and throughout this paper let X be a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers. A mapping $f: G \rightarrow X$ is said to be *quadratic* if and only if it satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in G.$$

A mapping $f: G \rightarrow X$ is said to be *additive* if and only if it satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in G$.

Let $G_i, i = 1, 2, \dots, n$ be an abelian groups and throughout this paper let X be a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers.

Definition 1.1. A mapping $F_k: G_1 \times G_2 \times \dots \times G_n \rightarrow X$ is called is additive mapping respect to k th variable if

$$F_k(x_1, \dots, a_k + b_k, \dots, x_n) - F_k(x_1, \dots, a_k, \dots, x_n) - F_k(x_1, \dots, b_k, \dots, x_n) = 0$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$).

Definition 1.2. For $\beta \neq 0, \alpha + \beta \leq 1$ and $s \in (0,1]$ a mapping $F_k: G_1 \times G_2 \times \dots \times G_n \rightarrow X$ is called is s -convex type respect to k th variable whit the following notation if

$$DF(a_k, b_k) := F_k(x_1, \dots, \alpha(a_k) + \beta(b_k), \dots, x_n) - \alpha^s F_k(x_1, \dots, a_k, \dots, x_n) - \beta^s F_k(x_1, \dots, b_k, \dots, x_n)$$

for all $a_k, b_k \in G_k$ and $x_i \in G_i$ ($i = 1, 2, \dots, n$).

for given sets $A, B \subseteq X$ and a number $k \in \mathbb{R}$, we define the well known operations

$$A + B := \{a + b: a \in A, b \in B\}, \quad kA := \{ka: a \in A\}.$$

We denote the convex hull of a set $U \subseteq X$ by $\text{conv}(U)$ and by $\bar{U}$ the sequential closure of U . Moreover it is well know that:

(1) If $A, B \subseteq X$ are bounded sets, then $A + B$ $\text{conv}(A)$ and $\bar{A}$ are bounded subsets of X .

(2) If $A, B \subseteq X$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha \text{conv}(A) + \beta \text{conv}(B) = \text{conv}(\alpha A + \beta B)$.

We define f is s, n -additive function if $f(x_1, \dots, r x_k, \dots, x_n) = r^s f(x_1, \dots, x_k, \dots, x_n)$, for all $x_i \in G_i$ ($i = 1, 2, \dots, n$) and all $r \in \mathbb{Q}$.

II. Main Results

We start with the following lemma.

Lemma 2.1. Let $G_1, \dots, G_n$ be a abelian groups and $B \subseteq X$ be a nonempty set. If the functions $F, H: G_1 \times G_2 \times \dots \times G_n \rightarrow X$ satisfy

$$F_k(x_1, \dots, \alpha a_k + \beta b_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n) \in B$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). Then

$$DF(a_k, b_k) \in \text{conv}(B) + 2\text{conv}(-B) - (\alpha^s + \beta^s)H_k(x_1, \dots, 0, \dots, x_n)$$

$$DH(a_k, b_k) \in \text{conv}(B) + \left(\frac{1}{\alpha}\right)^s (\beta^s H_k(x_1, \dots, 0, \dots, x_n) + \text{conv}(-B))$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$).

Proof. Putting $b_k = 0$ in (2.1), we get

$$F_k(x_1, \dots, \alpha a_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) \in \beta^s H_k(x_1, \dots, 0, \dots, x_n) + B$$

$$\alpha^s F_k(x_1, \dots, a_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) \in \beta^s H_k(x_1, \dots, 0, \dots, x_n) + B$$

for all $x \in G$. If we putting $a_k = 0$ in (2.1), then we have

$$F_k(x_1, \dots, \beta b_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n) \in \alpha^s H_k(x_1, \dots, 0, \dots, x_n) + B$$

$$\beta^s F_k(x_1, \dots, b_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n) \in \alpha^s H_k(x_1, \dots, 0, \dots, x_n) + B$$

for all $y \in G$. It follows from (2.5) and (2.1) that

$$DF(x, y) = F_k(x_1, \dots, \alpha a_k + \beta b_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n) - [\alpha^s F_k(x_1, \dots, a_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n)] - [\beta^s F_k(x_1, \dots, b_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n)] \in \text{conv}(B) + 2\text{conv}(-B) - (\alpha^s + \beta^s)H_k(x_1, \dots, 0, \dots, x_n)$$

Moreover, we have

$$DH(x, y) = F_k(x_1, \dots, \alpha a_k + \beta b_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) - \beta^s H_k(x_1, \dots, b_k, \dots, x_n) - [F_k(x_1, \dots, \alpha a_k + \beta b_k, \dots, x_n) - H_k(x_1, \dots, \alpha a_k + \beta b_k, \dots, x_n)] \in \text{conv}(B) + \left(\frac{1}{\alpha}\right)^s (\beta^s H_k(x_1, \dots, 0, \dots, x_n) + \text{conv}(-B))$$

Theorem 2.2. Let $G_1, \dots, G_n$ be an abelian 2-divisible groups and $B \subseteq X$ be a bounded set. Suppose that the functions $F, H: G_1 \times G_2 \times \dots \times G_n \rightarrow X$ satisfy (2.1) for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). Then there exists exactly one additive function $A: G_1 \times G_2 \times \dots \times G_n \rightarrow X$ such that

$$A(x_k) - F_k(x_1, \dots, a_k, \dots, x_n) \in \frac{\alpha^s}{\beta} \bar{\Phi}, \quad A(x_k) - H_k(x_1, \dots, a_k, \dots, x_n) \in \frac{\alpha^s}{\beta} \bar{\Phi} \quad (2.6)$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$) Moreover, the function A is given by

$$A(x_k) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} F_k(x_1, \dots, 2^n x_k, \dots, x_n) = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} H_k(x_1, \dots, 2^n x_k, \dots, x_n)$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). Moreover, the convergence of the sequences are uniform on

$$x_i \in G_i \quad (i = 1, 2, \dots, n)$$

Proof. By Lemma 2.1, we get (2.2). setting $b_k=0$ in (2.2), we get

$$F_k(x_1, \dots, \alpha a_k, \dots, x_n) - \alpha^s H_k(x_1, \dots, a_k, \dots, x_n) \in \beta^s H_k(x_1, \dots, 0, \dots, x_n) + B := \Phi, \quad (2.7)$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). Therefore by replacing a_k by $\frac{a_k}{\alpha^{s(n+1)}}$ in (2.7) and multiplying both sides (2.7) to α^{sn} we have

$$\alpha^{sn} F\left(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n\right) - \alpha^{s(n+1)} F\left(x_1, \dots, \frac{a_k}{\alpha^{s(n+1)}}, \dots, x_n\right) \in \Phi \quad (2.8)$$

SO

$$\begin{aligned} & \alpha^{sn} F\left(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n\right) - \alpha^{s(m+1)} F\left(x_1, \dots, \frac{a_k}{\alpha^{s(m+1)}}, \dots, x_n\right) \\ &= \sum_{j=m}^n [\alpha^{sj} F\left(x_1, \dots, \frac{a_k}{\alpha^{sj}}, \dots, x_n\right) - \alpha^{s(j+1)} F\left(x_1, \dots, \frac{a_k}{\alpha^{s(j+1)}}, \dots, x_n\right)] \\ &\in \sum_{j=m}^{n-1} \alpha^{sj} \Phi \\ &\subseteq \frac{\alpha^{s+m}}{\beta} \Phi \end{aligned} \quad (2.9)$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$) and all integers $n > m \geq -1$. Since B is bounded, we conclude that $\text{conv}(B - B)$ is bounded. It follows from (2.9) and boundedness of the set $\text{conv}(B - B)$ that the sequence $\{\alpha^{sn} F(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n)\}$ is (uniformly) Cauchy in X for all $x_i \in G_i$ ($i = 1, 2, \dots, n$) Since X is a sequential complete topological vector space, the sequence

$\{\alpha^{sn} F(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n)\}$ is convergent for all $x_i \in G_i$ ($i = 1, 2, \dots, n$) and the convergence is uniform on G . Define

$$A_1: G_1 \times G_2 \times \dots \times G_n \rightarrow X, \quad A_1(x_k) := \lim_{n \rightarrow \infty} \alpha^{sn} F\left(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n\right).$$

Since $\text{conv}(B - B)$ is bounded, let $a_k = b_k = x_k$, it follows from (2.2) that

$$D A_1(a_k, b_k) = \lim_{n \rightarrow \infty} \alpha^{sn} D F\left(\frac{a_k}{\alpha^{sn}}, \frac{b_k}{\alpha^{sn}}\right) = 0$$

for all $a_k, b_k \in G_k$. So $A_1(x_k)$ is additive (see [4]). Letting $m=-1$ and $n \rightarrow \infty$ in (2.9), we get

$$A_1(x_k) - F_k(x_1, \dots, a_k, \dots, x_n) \in \frac{\alpha^{s-1}}{\beta} \bar{\Phi} \quad (2.10)$$

for all $x_i \in G_k$ ($i = 1, 2, \dots, n$). Similarly as before applying (2.3) we have an additive mapping

$$A_2(x_k): G_1 \times G_2 \times \dots \times G_n \rightarrow X \text{ defined by } A_2(x_k) := \lim_{n \rightarrow \infty} \alpha^{sn} H(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n) \text{ which is satisfying}$$

$$A_2(x_k) - H_k(x_1, \dots, a_k, \dots, x_n) \in \frac{\alpha^{s-1}}{\beta} \bar{\Phi} \quad (2.11)$$

Where $\bar{\Phi} := \text{conv}(B) + (\frac{1}{\alpha})^s (\beta^s H_k(x_1, \dots, b_k, \dots, x_n) + \text{conv}(-B))$ for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). Since B is bounded, it follows from (2.5) that $A_1(x_k) = A_2(x_k)$. Letting $A(x_k) := A_1(x_k)$, we obtain (2.6) from (2.10) and (2.11).

To prove the uniqueness of $A(x_k)$, suppose that there exists another additive function

$A'(x_k): G_1 \times G_2 \times \dots \times G_n \rightarrow X$ satisfying (2.6). So

$$A'(x_k) - A(x_k) = \left[A'(x_k) - F\left(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n\right) \right] + \left[F\left(x_1, \dots, \frac{a_k}{\alpha^{sn}}, \dots, x_n\right) - A(x_k) \right] \in \frac{2\alpha^{s-1}}{\beta} \bar{\Phi}$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). since $A'(x_k)$ and $A(x_k)$ are additive, replacing a_k by $\frac{a_k}{\alpha^{sn+1}}$ implies that

$$A'(x_k) - A(x_k) \in \frac{2\alpha^{n+s-1}}{\beta} \bar{\Phi}$$

for all $x_i \in G_i$ ($i = 1, 2, \dots, n$). and all integers n . since $\bar{\Phi}$ is bounded, we infer $A' = A$. This completes the proof of theorem.

III. References

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