



η -DUALS OF GENERALIZED DIFFERENCE SEQUENCE SPACES

Shadab Ahmad Khan* and Ashfaq A. Ansari

Department of Mathematics and Statistics,

D. D. U. Gorakhpur University, Gorakhpur-273009, Uttar Pradesh, India

Abstract: Mikail Et. and R. Colak [1] have generalized difference sequence spaces. P. Chandra and B. C. Tripathy [9] have introduced the concept of η -duals. Kizmaz [2] introduce the difference sequence spaces. We have introduced the concepts of η -duals and determine η -duals of some difference sequence spaces.

Keywords: Difference sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ ($m \in \mathbb{N}$), η -duals, perfect spaces.

I. Introduction

Let ℓ_∞ , c , c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, \dots\}$, the set of positive integers.

Kizmaz [2] defined the sequence spaces

$$\ell_\infty(\Delta) = \{x = (x_k) : \Delta x \in \ell_\infty\},$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\}, \quad c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$\|x\| = \|x_1\| + \|\Delta x\|_\infty.$$

After then Et [6] defined the sequence spaces

$$\ell_\infty(\Delta^2) = \{x = (x_k) : \Delta^2 x \in \ell_\infty\},$$

$$c(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c\}, \quad c_0(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c_0\},$$

where $\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1})$, and showed that these are Banach spaces with norm

$$\|x\|_1 = \|x_1\| + \|x_2\| + \|\Delta^2 x\|_\infty.$$

M. Et and R. Colak [1] defined the sequence spaces

$$\ell_\infty(\Delta^m) = \{x = (x_k) : \Delta^m x \in \ell_\infty\},$$

$$c(\Delta^m) = \{x = (x_k) : \Delta^m x \in c\},$$

$$c_0(\Delta^m) = \{x = (x_k) : \Delta^m x \in c_0\}, \text{ where } m \in \mathbb{N},$$

$$\Delta^0 x = (x_k), \quad \Delta x = (x_k - x_{k+1}),$$

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}), \text{ and so that}$$

$$\Delta^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+v}.$$

It is trivial that $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ are linear spaces. It is also easy to show that these sequence spaces are normed spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^m |x_i| + \|\Delta^m x\|_\infty. \quad (1.1)$$

1. Main Results

The sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ are Banach spaces with norm (1.1)

Lemma 1.1.

$$(i) \quad c_0(\Delta^m) \subset c_0(\Delta^{m+1}),$$

$$(ii) \quad c(\Delta^m) \subset c(\Delta^{m+1}),$$

$$(iii) \quad \ell_\infty(\Delta^m) \subset \ell_\infty(\Delta^{m+1}),$$

and the inclusions are strict.

Proof. Let $x \in c_0(\Delta^m)$. Since

$$|\Delta^{m+1} x_k| = |\Delta^m x_k - \Delta^m x_{k+1}|$$

$$\leq |\Delta^m x_k| + |\Delta^m x_{k+1}| \rightarrow 0, (k \rightarrow \infty)$$

we obtain $x \in \Delta^{m+1}(c_0)$. Thus $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$. This inclusion is strict since the sequence $x = (k^m)$, for example, belongs to $c_0(\Delta^{m+1})$, but does not belong to $c_0(\Delta^m)$.

The proofs of (ii) and (iii) is similar to that of (i).

Lemma 1.2.

$$\begin{aligned} (i) \quad & c_0(\Delta^m) \subset c(\Delta^m), \\ (ii) \quad & c(\Delta^m) \subset \ell_\infty(\Delta^m), \end{aligned}$$

and the inclusions are strict.

The proof is similar to that of Lemma 1.1.

Note. $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ need not to be sequence algebra. We give a counterexample. It is well-known that c_0 is a sequence algebra.

Let $x = (k)$, $y = (k^{m-1})$. Clearly $x, y \in c_0(\Delta^m)$, $x \cdot y \notin c_0(\Delta^m)$ for $m \geq 2$.

Let us define the operator.

$$D : \ell_\infty(\Delta^m) \rightarrow \ell_\infty(\Delta^m)$$

as $Dx = (0, 0, \dots, x_{m+1}, x_{m+2}, \dots)$, where $x = (x_1, x_2, x_3, \dots)$.

It is trivial that D is a bounded linear operator on $\ell_\infty(\Delta^m)$. Furthermore the set

$D[\ell_\infty(\Delta^m)] = D\ell_\infty(\Delta^m) = \{x = (x_k) : x \in \ell_\infty(\Delta^m), x_1 = x_2 = \dots = x_m = 0\}$ is a subspace of

$\ell_\infty(\Delta^m)$, and $\|x\|_\Delta = \|\Delta^m x\|_\infty$ in $D\ell_\infty(\Delta^m)$.

Now let us define

$$\Delta^m : D\ell_\infty(\Delta^m) \rightarrow \ell_\infty, \Delta^m x = y = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}). \quad (1.2)$$

It can be shown that Δ^m is a linear homeomorphism. Hence $D\ell_\infty(\Delta^m)$ and ℓ_∞ are equivalent as topological spaces [7]. Δ^m and $(\Delta^m)^{-1}$ are norm preserving and $\|\Delta^m\| = \|(\Delta^m)^{-1}\| = 1$.

Let ℓ'_∞ and $[D\ell_\infty(\Delta^m)]'$ denote the continuous duals of ℓ_∞ and $D\ell_\infty(\Delta^m)$ respectively.

It can be shown that

$$\begin{aligned} T : [D\ell_\infty(\Delta^m)]' &\rightarrow \ell'_\infty \\ f_\Delta &\mapsto f_\Delta \circ (\Delta^m)^{-1} = f \end{aligned}$$

is a linear isometry. So $[D\ell_\infty(\Delta^m)]'$ is equivalent to ℓ'_∞ [7].

In the same way it can be shown that $Dc(\Delta^m)$ and c , $Dc_0(\Delta^m)$ and c_0 are equivalent as topological spaces and $[Dc(\Delta^m)]' \cong [Dc_0(\Delta^m)]' = \ell_1$, where

$$\ell_1 = \{x = (x_k) : \sum_{k=1}^\infty x_k < \infty\}.$$

2. Dual spaces of $\ell_\infty(\Delta^m)$ and $c(\Delta^m)$

In this section we give η -duals of $\ell_\infty(\Delta^m)$ and $c(\Delta^m)$. Also we show that these spaces are not perfect space relative to η -dual.

Lemma 2.1. $\sup_k k^{-1} |\Delta^{m-1}x_k| < \infty$ implies $\sup_k k^{-m} |x_k| < \infty$.

Lemma 2.2. $x \in \ell_\infty(\Delta^m)$ implies $\sup_k k^{-m} |x_k| < \infty$.

Definition 2.3. ([9]) Let X be a non-empty subset of ω (The space of all complex sequences) and $r \geq 1$. Then η -dual of X is define as.

$$X^\eta = \{a = (a_k) : \sum_{k=1}^\infty |a_k x_k|^r < \infty \text{ for all } x \in X\}$$

A non-empty subset X of ω is set to be perfect or η -reflexive if $X^{\eta\eta} = X$.

Taking $r = 1$ in above definition we get α -dual of X .

Theorem 2.4. $[D\ell_\infty(\Delta^m)]^\eta = \{a = (a_k) : \sum_{k=1}^\infty k^{mr} |a_k| < \infty\}$.

Proof. Let $U_1 = \{a = (a_k) : \sum_{k=1}^\infty k^{mr} |a_k| < \infty\}$.

If $a \in U_1$, then

$$\sum_{k=1}^\infty |a_k x_k|^r = \sum_{k=1}^\infty k^{mr} |a_k|^r (k^{-m} |x_k|)^r < \infty$$

for each $x \in D\ell_\infty(\Delta^m)$, by Lemma 2.2

Hence $a \in [D\ell_\infty(\Delta^m)]^\eta$.

Let $a \in [D\ell_\infty(\Delta^m)]^\eta$. Then $\sum_{k=1}^\infty |a_k x_k|^r < \infty$ for each $x \in D\ell_\infty(\Delta^m)$.

For the sequence $x = (x_k)$, defined by

$$x_k = \begin{cases} 0, & k \leq m \\ k^m, & k > m \end{cases} \quad (2.41)$$

We can write,

$$\sum_{k=1}^{\infty} k^{mr} |a_k|^r = \sum_{k=1}^m k^{mr} |a_k|^r + \sum_{k=1}^{\infty} |a_k x_k|^r < \infty.$$

This implies $a \in U_1$.

Theorem 2.5. $[D \ell_{\infty}(\Delta^m)]^{\eta} = [Dc(\Delta^m)]^{\eta}$

Proof. Since $Dc(\Delta^m) \subset D \ell_{\infty}(\Delta^m)$.

Therefore $[D \ell_{\infty}(\Delta^m)]^{\eta} \subset [Dc(\Delta^m)]^{\eta}$.

(2.51)

Let $a \in [Dc(\Delta^m)]^{\eta}$. Then $\sum_{k=1}^{\infty} |a_k x_k|^r < \infty$ for each $x \in Dc(\Delta^m)$.

If we take the sequence $x = (x_k)$, which is defined in (2.41), then we have

$$\sum_{k=1}^{\infty} k^{mr} |a_k|^r = \sum_{k=1}^m k^{mr} |a_k|^r + \sum_{k=1}^{\infty} |a_k x_k|^r < \infty.$$

This implies that $a \in [D \ell_{\infty}(\Delta^m)]^{\eta}$

Therefore $[Dc(\Delta^m)]^{\eta} \subset [D \ell_{\infty}(\Delta^m)]^{\eta}$

(2.52)

From (2.51) and (2.52) we get

$$[D \ell_{\infty}(\Delta^m)]^{\eta} = [Dc(\Delta^m)]^{\eta}$$

Theorem 2.6. $[\ell_{\infty}(\Delta^m)]^{\eta} = [D \ell_{\infty}(\Delta^m)]^{\eta}$

Proof. Since $D \ell_{\infty}(\Delta^m) \subset \ell_{\infty}(\Delta^m)$. Then

$$[\ell_{\infty}(\Delta^m)]^{\eta} \subset [D \ell_{\infty}(\Delta^m)]^{\eta}$$

(2.61)

Let $a \in [D \ell_{\infty}(\Delta^m)]^{\eta}$ and let $x \in \ell_{\infty}(\Delta^m)$.

From Lemma 2.2 we have

$$\sum_{k=1}^{\infty} |a_k x_k|^r = \sum_{k=1}^{\infty} k^{mr} |a_k|^r [k^{-mr} |x_k|^r] < \infty.$$

Hence $a \in [\ell_{\infty}(\Delta^m)]^{\eta}$.

Therefore $[D \ell_{\infty}(\Delta^m)]^{\eta} \subset [\ell_{\infty}(\Delta^m)]^{\eta}$

(2.62)

From (2.61) and (2.62) we have

$$[\ell_{\infty}(\Delta^m)]^{\eta} = [D \ell_{\infty}(\Delta^m)]^{\eta}$$

Theorem 2.7. $[c(\Delta^m)]^{\eta} = [Dc(\Delta^m)]^{\eta}$

Proof. Since $Dc(\Delta^m) \subset c(\Delta^m)$.

$$\therefore [c(\Delta^m)]^{\eta} \subset [Dc(\Delta^m)]^{\eta}$$

(2.71)

Let $a \in [Dc(\Delta^m)]^{\eta}$ and let $x \in c(\Delta^m)$.

From Lemma 2.2 we have

$$\sum_{k=1}^{\infty} |a_k x_k|^r = \sum_{k=1}^{\infty} k^{mr} |a_k|^r [k^{-mr} |x_k|^r] < \infty.$$

For $x \in c(\Delta^m) \subset \ell_{\infty}(\Delta^m)$.

Hence $a \in [c(\Delta^m)]^{\eta}$

Therefore $[Dc(\Delta^m)]^{\eta} \subset [c(\Delta^m)]^{\eta}$

(2.72)

From (2.71) and (2.72)

$$[c(\Delta^m)]^{\eta} = [Dc(\Delta^m)]^{\eta}.$$

Theorem 2.8. Let $X = \ell_{\infty}$ or c . Then

$$[X(\Delta^m)]^{\eta} = \{a = (a_k) : \sum_{k=1}^{\infty} k^{mr} |a_k|^r < \infty\}.$$

Proof. Proof follows from Theorem 2.5, 2.6 and 2.7.

Hence $[\ell_{\infty}(\Delta^m)]^{\eta} = [c(\Delta^m)]^{\eta}$.

Corollary 2.9. For $X = \ell_{\infty}$ or c , then

$$[X(\Delta)]^{\eta} = \{a = (a_k) : \sum_{k=1}^{\infty} k^r |a_k|^r < \infty\}.$$

Proof. Take $m=1$ in Theorem 2.8.

Corollary 2.10. For $X = \ell_{\infty}$ or c , then

$$[X(\Delta^2)]^{\eta} = \{a = (a_k) : \sum_{k=1}^{\infty} k^{2r} |a_k|^r < \infty\}.$$

Proof. Take $m=2$ in Theorem 2.8.

Hence $[\ell_{\infty}(\Delta^2)]^{\eta} = [c(\Delta^2)]^{\eta}$

Theorem 2.11. Let X stand for ℓ_{∞} or c . Then

$$[X(\Delta^m)]^{\eta\eta} = \{a = (a_k) : \sup_k k^{-mr} |a_k|^r < \infty\}.$$

Proof. Let $U_2 = \{a = (a_k) : \sup_k k^{-mr} |a_k|^r < \infty\}$.

If $a \in U_2$ and $x \in [X(\Delta^m)]^{\eta}$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k|^r &= \sum_{k=1}^{\infty} k^{mr} |x_k|^r k^{-mr} |a_k|^r \\ &\leq \sum_{k=1}^{\infty} k^{mr} |x_k|^r \sup_k k^{-mr} |a_k|^r < \infty. \end{aligned}$$

Hence $a \in [X(\Delta^m)]^{\eta\eta}$.

Now suppose that $a \in [X(\Delta^m)]^{\eta\eta}$ and $a \notin U_2$. Then we have

$$\sup_k k^{-mr} |a_k|^r = \infty.$$

Hence there is strictly increasing sequence $[k(i)]$ of positive integers $k(i)$ such that $|a_{k(i)}|^r > i^{mr} [k(i)]^{mr}$. We define the sequence x by.

$$x_k = \begin{cases} |a_{k(i)}|^{-1}, & k = k(i) \\ 0, & k \neq k(i) \end{cases}$$

Then we have

$$\sum_{k=1}^{\infty} k^{mr} |x_k|^r = \sum_{i=1}^{\infty} [k(i)]^{mr} |a_{k(i)}|^{-r} \leq \sum_{i=1}^{\infty} i^{-mr} < \infty.$$

Hence $x \in [X(\Delta^m)]^\eta$ and $\sum_{k=1}^{\infty} |a_k x_k|^r = \sum 1 = \infty$.

This is contradiction to a $\in [X(\Delta^m)]^{\eta\eta}$. Hence $a \in U_2$.

Therefore $[\ell_\infty(\Delta^m)]^{\eta\eta} = c(\Delta^m)^{\eta\eta}$.

Corollary 2.12. For $X = \ell_\infty$ or c , then

$$[X(\Delta^2)]^{\eta\eta} = \{a = (a_k) : \sup_k k^{-2r} |a_k|^r < \infty\}.$$

Proof. Take $m = 2$ in Theorem 2.11, we have

$$[\ell_\infty(\Delta^2)]^{\eta\eta} = c(\Delta^2)^{\eta\eta}.$$

Corollary 2.13. The sequence spaces $\ell_\infty(\Delta^m)$ and $c(\Delta^m)$ are not perfect space relative to η -dual.

Proof. Proof is trivial as $X^{\eta\eta} \neq X$ for $X = \ell_\infty(\Delta^m)$ or $c(\Delta^m)$.

Acknowledgements

The author would like to express their deep sense of gratitude to Department of Mathematics and Statistics, D. D. U. Gorakhpur University, Gorakhpur-273009 (India), providing all necessary resources for research.

References

- [1] M. Et and R. Colak, On Some Generalized Difference Sequence Spaces Soochow Journal of Mathematics 21 (1995), 377-386.
- [2] H. Kizmaz, On Certain Sequence Spaces, Canad. Math. Bull 24 (1981), 169-176.
- [3] P. K. Kamphth and Gupta, Sequence Spaces and Series, Marcel Dekker Inc. New York, 1981.
- [4] R. Colak, On Some Generalized Difference Sequence spaces, Commun. Fac. Sci Univ. Ank. Serise A₁, 38 (1989), 35-46.
- [5] R. Colak, P. D. Srivastva and S. Nanda, On Certain Sequence Spaces and there Köthe-Toeplitz Duals, Rendiconti di Mat.13 (1993), 18-24.
- [6] M. Et On Some Difference Sequence Spaces, Doga- Tr.J of Mathematics, 17 (1993), 18-24.
- [7] I. J. Maddox, Elements of Functional Analysis Cambridge Univ. Press, 1970.
- [8] M. Et. and A. Esi, On Köthe -Toeplitz Duals, of Generalized Difference Sequence Spaces, Bull. Malaysian Math. Sc. Soc., 23(2000), 25-32.
- [9] P. Chandra and B. C. Tripathy, On Generalized Köthe-Toeplitz Duals of Some Sequence Spaces, Indian J. Pure appl. Math., 33 (8) 2002, 1301- 1306.
- [10] R. G. Cooke, Infinite Matrices and Sequence Spaces, Dover Publ., 1955.