



## Cooperative Fuzzy Game Theoretic Approach to Multi-Objective Linear Fractional Programming Problems

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**Abstract:** Game theoretic approaches to cooperative situations in fuzzy environments have given rise to several types of cooperative fuzzy games. Solving multi-objective decision-making problems is, generally, a very difficult goal. In these particular optimization problems, the objectives often conflict across a high-dimensional problem space and may also require extensive computational resources. The cooperative game theory and fuzzy set theory together give a new optimization technique namely, cooperative fuzzy game, which can be applied to solve not only well and ill-structured single and multiple objective problem, but even the preliminary decision making and other problems where only a feasible solution is sought with no objective function being specified. In this paper, we apply cooperative game theoretic approach to solve some multi-objective linear fractional programming problems. This has been done by introducing the concept of utility function, Pareto optimality, and the fuzzy set theory. It is shown that the method for solving crisp and fuzzy optimization problem are special cases of the fuzzy game theoretical formulation. Our study concerns cooperative games with fuzzy coalitions. A numerical example is solved to illustrate the potential use of the approach.

**Keywords:** Cooperative game theory, Fuzzy programming, Linear membership function, Linear fractional programming problem, Multi-objective decision making, Optimal solution, Satisfactory solution.

### I. INTRODUCTION

In this paper we deal with multi-objective linear fractional programming problems (MOLFPP) with the essentially co-operative decision makers (DMs) and propose a solution procedure using cooperative game theoretic approach to solve MOLFPP. Most of the real world optimization problems naturally involve multiple objectives. Different solutions may produce different trade-offs (conflicting scenarios) among different objectives. A solution that is extreme (in a better sense) with respect to one objective requires compromise in other objectives. In problems with more than one conflicting objective there exists no single optimal solution but a number of solutions all of which are optimal. Such solutions are called Pareto optimal solutions. Usually the user is interested in finding all the Pareto optimal solutions.

Game theory is the discipline which studies multiple individuals implementing the corresponding strategy according to related strategies of other individuals under some situations. Sometimes we need to seek for the best strategies of each player taking into account that the others will also behave searching for their best. In this case, we call it as non-cooperative games model. On the other hand, players just want to deal with the cooperation issues of the problem and consider how the agents allocate the benefits of their cooperation. This approach is called as the cooperative game [24,25]. A cooperative game often assumes that each player is a part of a team and is willing to compromise his own payoff to improve the goal as a whole. A cooperative game proceeds with the intent that the team wants to allocate resources such that all players are as better off as possible, and an improvement in the payoff for one player does not result in a loss for other players. The bargaining scheme postulated by Nash yields a unique and optimal distribution of resources such that the arbitrated outcome is Pareto-optimal [24,25]. Classical game theory is based on binary logic and the fully rational behaviour assumption. Fuzzy logic is able to accommodate many of the binary-logic related dilemmas in classical game theory. In general, the players do not be having as fully rational decision makers in real games. Fuzzy logic is a tool for a formal representation of such behaviour.

The goal of this study is to establish a flexible fuzzy solution approach for solving MOLFPP using game theory in co-operative environment. The main advantage of the proposed approach is that the possibility of rejecting the solution again and again by the fuzzy approach and re-evaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach to the satisfactory decision does not arise. Moreover, one of the outstanding limitations of the classical game theory is that it assumes that all the data are known exactly by all players. This assumption is often restrictive. In real world, it often happens that the players are not able to

evaluate exactly the outcomes of different strategy profiles and their own preferences or the preferences of other players [20]. Therefore, to characterize the bounded rational behaviour and games with imperfect or incomplete information, it is necessary to employ fuzzy logic into the game theory. Fuzzy logic was initiated by Zadeh [37] for dealing with uncertainties. From then on, fuzzy theory was extensively applied in many areas, such as decision sciences [4], control theory and games theory [2, 8, 20]. Aubin [1] first studied the problem of fuzzy cooperative games. Dhingra and Rao [8] integrated the cooperative game theory and fuzzy set theory to yield a new optimization method.

Fractional programming (FP) which has been being used as an important planning tool for the past four decades is applied for a lot of disciplines such as engineering, business, finance, economics etc. FP is generally used for modeling real life problem which has one or more than one objective(s) as a ratio of two functions such as profit/loss, inventory/sales, actual cost/standard cost, output/employee etc. Fractional programs arise in various contexts such as, in investment problems, the firm wants to select a number of projects on which money is to be invested so that the ratio of the profits to the capital invested is maximum subject to the total capital available and other economic requirements which may be assumed to be linear. If the price per unit depends linearly on the output and the capital is a linear function then the problem is reduced to a linear fractional program.

An example of linear fractional programming was first identified and solved by Isbell and Marlow[12]. Their algorithm generates a sequence of linear programs whose solutions converge to the solution of the fractional program in a finite number of iterations. Since then several methods of solutions were developed. Gilmore and Gomory [10] modified the simplex method to obtain a direct solution of the problem. Martos [23] has suggested a simplex-line procedure, while by making a transformation of variables, Charnes and Cooper[5] have shown that a solution of the problem can be obtained by solving at most two ordinary linear programs. Algorithms based on the parametric form of the problem have been developed by Jagannathan [15] and Dinkelbach [7].

After the development of the method by Isbell and Marlow [12] for solving linear fractional programming problems, various aspects of single objective mathematical programming have been studied quite extensively. It was however realized that almost every real-life problem involves more than one objective. For such problems, the decision makers have to deal with several objectives conflicting with one another, which are to be optimized simultaneously. For example, in transportation problem, one might like to minimize the operating cost, minimize the average shipping time, minimize the production cost and maximize its capacity. Similarly, in production planning, the plant manager might be interested in obtaining a production programme which would simultaneously maximize profit, minimize the inventory of the finished goods, minimize the overtime and minimize the back orders. Several other problems in modern management can also be identified as having multiple conflicting objectives i.e., multi-objective decision-making (MODM) problem. There is pressing need to develop approaches to solve such type of multi-objective linear or non-linear fractional programming problems.

Multi-objective linear fractional programming (MOLFP) problems are studied by a few approaches which have appeared in [16,18, 22,31]. Luhandjula [22] proposed a linguistic approach to multi objective linear fractional programming by introducing linguistic variables to represent linguistic aspirations of the decision makers. The model of the problem constructed with fuzzy data due to FP approaches [22,31] used to solve MOLFP problems have a difficulty in computation. In the framework of fuzzy decision, Bellman and Zadeh [4], Sakawa and Yumine [31] presented a fuzzy programming approach for solving multi objective linear fractional programming problem by combined use of the bi-section method and the phase one of simplex methods of linear programming. Multi-level multi-objective linear or non-linear programming problems are new combination problems in the field of multi-level (or multi- objective) decision making problems. Ibrahim [14] proposed Fuzzy goal programming algorithm for solving decentralized bi-level multi-objective programming problems. Again, Ibrahim [13] proposed a fuzzy goal programming approach to solve multi-level multi-objective linear programming problems. Eren and Turan [9] used Fuzzy multi-objective linear programming approach for optimizing a closed-loop supply chain network. Again for MOLFP, Kornbluth and Steuer [16] presented two different approaches to MOLFP, based on the weighted Tchebyscheff norm.

In this paper, the cooperative fuzzy game theoretic approach [30] will be extended to solve MOLFP problems with the essentially cooperative DMs. The basis of cooperative game theory was laid in the book of von Neumann and Morgenstern [36]. Since then several solution concepts for cooperative games have been proposed and several interesting subclasses of games have been introduced. The theory of cooperative fuzzy games started with work of Aubin [1,2] where the notions of fuzzy game and the core of a fuzzy game are introduced. In the meantime many solution concepts have been developed.

## II. MULTI OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEMS AND VECTOR MINIMIZATION PROBLEM

A MOLFP problem contains more than one linear fractional objective function and linear or non-linear constraints. Mathematically, a MOLFP problem can be stated as:

$Min/Max(f_1, f_2, \dots, f_p);$

where each  $f_i : R^n \rightarrow R$ ; is linear fractional objective functions,

subject to

$$\sum_{j=1}^n a_{ij}x_j (\leq, \geq, =) b_i, \quad i = 1, 2, \dots, m$$

$$\bar{x} \geq 0.$$

Where objective functions  $f_i(x), i = 1, 2, \dots, m$  are represented by a linear fractional function

$$f_i(x) = \frac{p_i(x)}{q_i(x)}$$

Where,  $p_i(x)$  and  $q_i(x)$  are linear functions.  $\bar{x}$  is an  $n$  – dimensional vector.

The problem indicates that there are  $P$  numbers of objective functions, which are to be maximized or minimized. The problem contains  $m$  number of constraints and  $n$  number of decision variables.

Most real-world decision problems involve multiple criteria that are often conflict in general and it is sometimes necessary to conduct trade-off analysis in MODM. We can find many instances of decision problem, which are formulated as MODM problem, and in decentralized firm, it is natural that the decision makers behave cooperatively rather than non-cooperatively.

Each objective function is a function of  $n$  decision variables and each function has to be simultaneously optimized subject to the common constraints. Usually all the objective functions are conflicting in nature. Multi objective approach is necessary for the solution of integrated system.

A general vector minimization problem is of the form

$$Min: f(X) = [f_1(X), f_2(X), \dots, f_k(X)]^T$$

Subject to

$$X \in S = \{X / X \in R^n, g_i(X) \leq a_i, h_i(X) \leq b_i\}$$

$$i = 1, 2, \dots, p. \quad j = 1, 2, \dots, s.$$

(1)

Where,  $X$  is an  $n$  – dimensional vector of decision variables,  $f_1(X), f_2(X), \dots, f_k(X)$  are  $k$  distinct objective functions of  $k$  players and  $S$  is the set of feasible solutions. For a single objective problem, an optimum solution is define as one that minimizes the objective function  $f_i(X)$ ,  $X \in S$ . For a vector minimization problem, an attempt to define a vector minimal point as one at which all components of the objective function  $f$  are simultaneously minimized is not adequate generalization since such utopia points are seldom attainable. Therefore the concept of Pareto optimal and weakly Pareto optimal solutions [34] have been found to be useful in the context of vector minimization problems. A feasible solution  $X^* \in S$  is called a Pareto optimal solution if there is no  $\bar{X} \in S$  such that  $f_i(\bar{X}) \leq f_i(X^*)$ ,  $i = \{1, 2, \dots, k\}$  and  $f_j(\bar{X}) < f_j(X^*)$  for at least one i.e.,  $j \in \{1, 2, \dots, k\}$ . Alternately, a decision vector  $X^*$  is Pareto optimal if there exist no feasible vector  $\bar{X}$  which would decrease some objective function  $f_i$  without causing a simultaneous increase in at least one other objective  $f_k$ ,  $i \neq k$ . A feasible solution  $X^* \in S$  is called a weakly Pareto-optimal, if there is no  $\bar{X} \in S$  such  $f_i(\bar{X}) < f_i(X^*)$ ,  $i = \{1, 2, \dots, k\}$ . If  $X^*$  is Pareto-optimal then  $X^*$  is also weakly Pareto-optimal. However, the converse is true only under special circumstances. A game theoretic formulation for a vector minimization problems that helps determine an optimal compromise solution [34] from the set of Pareto-optimal solution is given next.

### III. COOPERATIVE GAME

A cooperative game [1,8,26,30] model assumes that each player is a part of the team and is willing to compromise his own payoff to improve the situation as a whole. A cooperative game proceeds with the intent that the team wants to allocate resources such that all players are as better off as possible, and an improvement in the payoff for one player does not result in an unacceptable loss for another player.

Consider a game-theoretic problem with two players. Assume that there exists a utility (payoff) function [11,30,35]  $U_i(X)$  associated with each player  $i$  such that if the strategy  $X \in S$  is selected, player  $i$ 's payoff will be  $U_i(X)$ . The two players engage in bargaining and desire a trade such that their utility functions are maximized. The bargaining model used in this work is as follows:

Assume that there exist a convex, bounded and closed set of alternatives  $S$ , a special point  $X_w \in S$  corresponding to point of no agreement between the players and a set of utility functions  $U_i(X)$ ,  $i = \{1, 2, \dots, k\}$  associated with each player. To determine a fair solution, a bargaining function  $B(X)$  is defined as:

$$B(X) = \prod_{i=1}^k [U_i(X) - U_i(X_w)] \quad (2)$$

for all  $X \in S^* \subset S$ , where

$$S^* = \{X : X \in S, U_i(X) - U_i(X_w) \geq 0\}, \quad i = 1, 2, \dots, k. \quad (3)$$

Consider now a multiple objective optimization problem with  $k$  objective functions (1). A game-theoretic formulation for this problem consists of  $k$  players where each player corresponds to an objective function. The bargaining function  $B(X)$  for this VMP problem given as:

$$B(X) = \prod_{i=1}^k [f_i(X_w) - f_i(X)]. \quad (4)$$

Here objective functions  $f_i$  have to be minimized where as, the utility function  $U_i(X)$ , in equation (2) are to be maximized. It is assumed here that all objective functions  $f_i$ 's are equally important. If we associate different degree of weights  $w_i$  with objective functions, the bargaining function  $B(X)$  is generalized as:

$$B(X) = \prod_{i=1}^k [f_i(X_w) - f_i(X)]^{w_i} \quad (5)$$

The weights  $w_i$  can be determined using the Saaty's method of pairwise comparisons [32,33]. Finally a game theoretic formulation for a VMP problem, yields,

$$\max : B(X) = \prod_{i=1}^k [f_i(X_w) - f_i(X)]^{w_i}$$

Subject to

$$X \in S^* \quad (6)$$

The game theoretic formulation for VMP problems (6): (i) yields an optimum solution that is Pareto-optimal, and (ii) ensures that at the final solution all objective functions attain acceptable values [8]. A generalization of

this formulation to the case when the problem data; constraints and/or objective functions are not known precisely is given next. The generalization is based on fuzzy set theory. Now, we will present some definitions which form the basis of a decision making-process in a fuzzy environment [30].

#### IV. COOPERATIVE FUZZY GAME THEORETIC APPROACH TO MOLFPs

The Von Neumann Morgenstern theory for cooperative games yields a multitude of such solutions that are all non-dominated. Here we first show how fuzzy programming approach rules over vector minimization problem and game theory may be used as a decision making tool in solving multi objective linear fractional programming problems. The cooperative game theory and fuzzy set theory are then combined to yield a new optimization method referred to here in as, cooperative fuzzy games. New fuzzy operators [39] have been introduced for aggregating membership functions corresponding to fuzzy objective functions. A completely general information capable of solving optimization problems with partially crisp and partly fuzzy objective functions, as well as partly crisp and partly fuzzy constraints is presented.

Consider two fuzzy sets  $A$  and  $B$  whose membership functions are  $\mu_1$  and  $\mu_2$

( $0 \leq \mu_1 \leq 1, 0 \leq \mu_2 \leq 1$ ). A fuzzy intersection  $i$  is defined as the mapping [17,30,39]:

$$i : [0,1] \times [0,1] \rightarrow [0,1] \quad (7)$$

with  $\mu_{(A \cap B)}(X) = i[\mu_A(\alpha), \mu_B(\alpha)]$  where  $\mu_A(\alpha)$  and  $\mu_B(\alpha)$  denotes the membership value of some element  $X$  in fuzzy sets  $A$  and  $B$  respectively and  $\times$  in equation (7), denotes the cartesian product of two fuzzy sets. For  $n$  fuzzy sets, the above definition can be generalized to yield  $i : [0,1]^n \rightarrow [0,1]$  where,  $[0,1]^n$  denotes the cartesian product of  $n$  fuzzy sets.

Next, consider a fuzzy optimization problem with fuzzy goals  $G_k, k = \{1, 2, \dots, n\}$  where  $G_k$  could be a fuzzy objective function or a fuzzy constraints. These fuzzy goals need to be instructed to determine an overall fuzzy decision function  $D$ . From a game theoretic point of view, when a min operator is employed to aggregate fuzzy sets, the decision process corresponds to a non-cooperative game [3] where each player is looking for his own interests and is unconcerned about the achievement level of other players. This in turns implies that an improvement in the satisfaction level of each player is at the expense of his opponents, and no cooperation exists between various players. On the other hand, the bargaining model permits a tradeoff between various goals whereby players improve the situation as a whole. The unsuitability of the min operator for aggregating fuzzy goals in a cooperative decision making process requires us to search for more appropriate fuzzy aggregation operators. A general class of means described next realizes a tradeoff between conflicting fuzzy goals by allowing positive comparison between various fuzzy ratings. A tradeoff in a decision making process corresponds to viewing the global evaluation of an action lying between the worst and the best individual ratings. This occurs in the presence of conflicting goals when a compromise between the corresponding compatibilities is allowed. A compromise attribute between conflicting goals is expressed as, for all  $a, b \in [0, 1]$ .

$$\min(a, b) < h(a, b) < \max(a, b)$$

Where,  $h$  is a suitable fuzzy aggregation operator that models a tradeoff between two fuzzy goals where membership values are  $a$  and  $b$  respectively. A class of averaging operators are called the generalized means.

We consider next a completely general version of the 'crisp' VMP problem given by equation (1). The generalized version has  $n$  fuzzy objective functions,  $m$  crisp objective functions,  $p$  fuzzy inequality constraints,  $r$  crisp inequality constraints,  $s$  fuzzy equality constraints and  $l$  crisp equality constraints. A linear membership function of a fuzzy objective function can be constructed as:

$$\mu_{\tilde{F}_i}(x) = \begin{cases} 0 & , & f_i(X) \geq f_i^{\max} \\ \frac{f_i^{\max} - f_i(X)}{f_i^{\max} - f_i^{\min}} & , & f_i^{\min} < f_i(X) < f_i^{\max} \\ 1 & , & f_i(X) \leq f_i^{\min} \end{cases} \quad (8)$$

where,  $f_i^{\min} = \min_j f_i(X_j^*)$  and  $F_{iu} = f_i^{\max} = \max_j f_i(X_j^*)$ ,  $X_j^*$  is the optimum design vector of the  $j$ -th objective function. The geometric mean captures the essence of the bargaining model in a cooperative decision making process by permitting a tradeoff between conflicting fuzzy goals. Aggregating  $n$  fuzzy sets using the geometric mean operation yields

$$\mu_{\bar{D}} = h(\mu_{F_1}, \mu_{F_2}, \dots, \mu_{F_n}) = \left[ \prod_{i=1}^n \mu_{F_i} \right]^{\frac{1}{n}} \quad (9)$$

For fuzzy inequality constraints stated as:

$$g_j(X) \leq a_j + c_j, \quad j = 1, 2, \dots, p \quad (10)$$

Where,  $c_j$  denotes the distance by which the boundary of the  $j$ -th constraints is moved, membership function for the  $j$ -th inequality constraints is constructed as:

$$\mu_{\tilde{G}_j}(x) = \begin{cases} 0 & , & g_j(X) \geq a_j + c_j \\ 1 - \frac{g_j(X) - a_j}{c_j} & , & a_j < g_j(X) < a_j + c_j, \quad j = 1, 2, \dots, p \\ 1 & , & g_j(X) \leq a_j \end{cases} \quad (11)$$

For fuzzy equality constraints stated as

$$h_j(X) \leq b_j + d_j, \quad j = 1, 2, \dots, s \quad (12)$$

Where,  $d_j$  denotes the distance by which the boundary of the  $j$ -th constraint is moved, the linear membership function of the  $j$ -th equality constraint is constructed as:

$$\mu_{\tilde{H}_j}(x) = \begin{cases} 0 & , & h_j(X) \geq b_j + d_j \\ 1 - \frac{h_j(x) - b_j}{d_j} & , & b_j - d_j < h_j(X) < b_j + d_j, \quad j = 1, 2, \dots, s \\ 0 & , & h_j(X) \leq b_j - d_j \end{cases} \quad (13)$$

Once the membership function of the fuzzy objectives and the fuzzy inequality and equality constraints i.e.,  $\mu_{F_j}^-$ ,  $\mu_{G_j}^-$  and  $\mu_{H_j}^-$  are known, then we construct the following mathematical programming problem:

Max  $B + a\lambda$

Subject to

$$\lambda \leq \mu_{F_j}^-, \quad \lambda \leq \mu_{G_i}^-, i = 1, 2, \dots, p, \quad \lambda \leq \mu_{H_j}^-, j = 1, 2, \dots, s. \quad (14)$$

$$X \in S = [X : X \in R^n, g_i \leq a_i, h_j = b_j, i = 1, 2, \dots, r, j = 1, 2, \dots, t.]$$

Where  $B$  and  $\mu_{F_j}^-$  are defined as :

$$B = \prod_{i=1}^m [F_{iu} - f_i] \quad (15)$$

$$\mu_{F_j}^- = \left[ \prod_{i=1}^n \mu_{F_i}^- \right]^{\frac{1}{n}} \quad (16)$$

and ' $a$ ' is a parameter supplied by the decision maker.

A (locally) Pareto-optimal solution to this generalized multi objective problem is obtained by solving the above problem which combines fuzzy and game theoretic concepts. The crisp objective functions are combined using the bargaining model given by equation (15). In this equation  $F_{iu}$  denotes the worst value for the  $i$ -th objective

function, which can be determined from (i) by finding solutions to individual single objective optimization problems:  $\min f_i(X)$  subject to the complete constraint set,  $i = 1, 2, \dots, (m+n)$  and (ii) by determining the best  $f_i(X_j^*)$  and worst values  $f_i(X_j^-)$ ,  $i, j = 1, 2, \dots, (m+n)$  for each of the objective functions. This computational procedure helps each player to analyze the maximum benefit he can obtain while negotiating with other players. It also appraises each player  $i$  of (i) the extreme conditions of other players  $[f_i(X_j^*), F_{iu}]$ , (ii) helps him in guaranteeing against his worst value ( $F_{iu}$ ) and (iii) ensures that none of the players begin bargaining from a reference value that is unacceptable to other players. The bar over a symbol shows that the expression or variable contains fuzzy information. The membership functions corresponding to fuzzy equality  $(\mu_{H_i}^-)$  and the inequality  $(\mu_{G_i}^-)$  constraints are aggregated using the fuzzy min operator. The min operator does not permit any tradeoff between the fuzzy constraints, which must all be satisfied during the optimization process. A min operator is used to aggregate fuzzy sets corresponding to fuzzy objective and constraint functions since the complete optimization problem entails that both fuzzy objective functions and fuzzy constraints functions be satisfied. The crisp equality and inequality constraints are appended to the problem formulation as given by equation (14). Let us consider the original fuzzy game theory problem (14) where there is no crisp constraints. An optimal solution to this problem is obtained by:

$$\left. \begin{array}{l} \max \quad B + a\lambda \\ \text{Subject to} \\ \lambda \leq \mu_{F_i}^-, \quad \lambda \leq \mu_{G_i}^-, i = 1, 2, \dots, p, \lambda \leq \mu_{H_j}^-, j = 1, 2, \dots, s. \end{array} \right\} \quad (17)$$

Consider the crisp VMP. In the weighting method  $F(x) = \sum_{i=1}^p w_i f_i$  is minimized. The Pareto-optimal solution is obtained using the weighting method depending on the weighting factor  $w_i$ . A similar trend is observed for the above problem (17) where the fuzzy Pareto-optimal depends on the weighting parameter ' $a$ '. By varying the value of ' $a$ ', a series of fuzzy Pareto-optimal solutions [8] can be obtained.

## V. STEPS OF PROPOSED COOPERATIVE FUZZY GAME THEORETIC APPROACH ARE AS FOLLOWS

**Step-1:** Establishing original linear fractional multi-objective optimization model.

**Step-2:** Setting payoffs as strategies of players.

**Step-3:** Fuzzifying objectives by membership functions.

**Step-4:** Making decision by Bellman and Zadeh's approach[4].

**Step-5:** Checking equilibrium (if not, go back to Step 1) and repeat step-1, step-2, step-3, step-4 and step-5.

**Step-6** Report the best compromise solution as the optimal solution.

## VI. NUMERICAL EXAMPLE

Multi-objective linear fractional programming problem.

$$\begin{aligned} \min \tilde{f}_1 &= \frac{-45x_1 - 51x_2 - 150x_3 - 250x_4}{x_1 + 5x_2 + 7x_3 + 3x_4} \\ \min \tilde{f}_2 &= \frac{-52x_1 - 54x_2 - 283x_3 - 489x_4}{4x_1 + 5x_2 + 6x_3 + 3x_4} \\ g_1(x) &= x_1 + x_2 + x_3 + x_4 \leq \tilde{15} \\ g_2(x) &= 7x_1 + 5x_2 + 3x_3 + 2x_4 \leq \tilde{120} \\ g_3(x) &= 3x_1 + 5x_2 + 10x_3 + 5x_4 \leq \tilde{100} \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned} \quad (18)$$

The two fuzzy objective functions with their goals as  $-63.33$ ,  $-113$  and the corresponding tolerance as  $20$ ,  $50$  respectively, which are given initially, and also are for the fuzzy resources:  $15$ ,  $120$ ,  $100$  and its corresponding tolerance  $5$ ,  $24$ , and  $20$  respectively. In the fuzzy set theory, the fuzzy objective constraints are defined by their corresponding membership functions. For simplicity, let us assume that the membership function  $\mu_{f_1}, \mu_{f_2}$  of

the fuzzy objectives and  $\mu_{g_i}, i = 1, 2, 3$  of the fuzzy constraints are non decreasing continuous linear functions.

Thus, we construct all the membership functions as follows:

$$\mu_{\tilde{f}_1}(x) = \begin{cases} 0 & , & -63 < f_1(x) \\ \frac{f_1(x) - 63}{20} & , & -83.33 < f_1(x) \leq -63 \\ 1 & , & f_1(x) \leq -83.33 \end{cases} \quad (19)$$

$$\mu_{\tilde{f}_2}(x) = \begin{cases} 0 & , & -113 < f_2(x) \\ \frac{-f_2(x) - 113}{50} & , & -163 \leq f_2(x) \leq -113 \\ 1 & , & f_2(x) \leq -163 \end{cases} \quad (20)$$

$$\mu_{\tilde{g}_1}(x) = \begin{cases} 1 & , & g_1(x) < 15 \\ 1 - \frac{g_1(x) - 15}{3} & , & 15 \leq g_1(x) \leq 18 \\ 0 & , & g_1(x) > 18 \end{cases} \quad (21)$$

$$\mu_{\tilde{g}_2}(x) = \begin{cases} 1 & , & g_2(x) < 120 \\ 1 - \frac{g_2(x) - 120}{24} & , & 120 \leq g_2(x) \leq 144 \\ 0 & , & g_2(x) > 120 \end{cases} \quad (22)$$

$$\mu_{\tilde{g}_3}(x) = \begin{cases} 1 & , & g_3(x) < 100 \\ 1 - \frac{g_3(x) - 100}{20} & , & 100 \leq g_3(x) \leq 120 \\ 0 & , & g_3(x) > 120 \end{cases} \quad (23)$$

Now, we formulate the fuzzy problem with the help of (17) as follows:

$$\max[-83.33 + f_1(x)][-163 + f_2(x)] + a\lambda$$

$$\lambda \leq \left[ \frac{f_1(x) - 63}{20} \right]^{\frac{1}{2}} \left[ \frac{f_2(x) - 113}{50} \right]^{\frac{1}{2}}$$

$$\lambda \leq \left[ 1 - \frac{x_1 + x_2 + x_3 + x_4 - 155}{3} \right]$$

$$\lambda \leq \left[ 1 - \frac{3x_1 + x_2 + 10x_3 + 15x_4 - 100}{20} \right]$$

$$\lambda \leq \left[ 1 - \frac{7x_1 + 5x_2 + 3x_3 + 2x_4 - 120}{24} \right]$$

$$x_1, x_2, x_3, x_4, \lambda \geq 0.$$

The solution of above (24) is obtained using LINGO software for different values of  $a$  and is shown in the following table:

Table

| $a$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\lambda$ | Objective function value | $f_1$   | $f_2$    |
|-----|-------|-------|-------|-------|-----------|--------------------------|---------|----------|
| 0   | 3.5   | 3.8   | 3.6   | 3.6   | 0         | -219.684                 | -103.94 | -103.93  |
| 1   | 3.38  | 4.4   | 2.9   | 2.7   | 0         | 321.790                  | -92.28  | -103.99  |
| 2   | 6.67  | 3     | 2.8   | 0.7   | 0         | 1116.320                 | -74.99  | -103.982 |
| 3   | 3.4   | 3     | 2.9   | 0.7   | 0         | 1116.330                 | -74.99  | -103.984 |
| 4   | 3.5   | 4.5   | 3.12  | 2.5   | 0         | 444.736                  | -89.60  | -103.98  |

## VII. DISCUSSION

The compromise to accept the uncertainty into the mathematical models must be done working with complex systems. In order to build and to solve real life problem it is natural to interpret the information in fuzzy manner. The philosophy of fuzzy sets theory is related to the model of human thinking and making decisions. Concepts of fuzzy sets theory "crowded" into a lot of research fields since 1980, when fuzzy logic has had a great success in the control systems theory.

Multi-objective linear fractional decision making problem is an important branch of Operation Research. Presently many approaches have been made to solve MODM problems mostly based on fuzzy programming. However, it has not yet made a significant breakthrough in the field of MOLFP problems. It is well known that the mathematical modelling of various real-world decision-making situations gives rise to combinatorial optimization problems. For situations where more than one decision-maker is involved classical combinatorial optimization theory does not suffice and it is here that cooperative game theory can make an important contribution. If a group of decision-makers decide to undertake a project together in order to increase the total revenue or decrease the total costs, they face several problems. In fuzzy game players are allowed to cooperate with infinitely many different participation levels, varying from non-cooperation to full cooperation. A multi-choice game describes an intermediate case in which each player may have a fixed number of activity levels.

Following are the advantages and special features of the proposed algorithm to solve MOLFP problems:

- (1) Multi-objective programming problems have some properties that are more complex than in the usual mathematical programs. Thus solution techniques for this problem have to be rather specialized. Solution techniques derived in the MODM literature often assume uniqueness, which is what is done in the exposition of this paper as well.
- (2) Traditional multi-objective programming methods can generate many candidate solutions, expanding a wide range of the objectives. Results are possibly difficult for the DMs to select which alternative. Results of the multi-objective method can only be utilized for the DMs without any prejudice about the situations. The DMs served as players who need to negotiate each other and have a series of concessions. In fuzzy game players are allowed to cooperate with infinitely many different participation levels, varying from non-cooperation to full cooperation. This helps proposed approach to explore a new, perhaps better part of the feasible region than previously considered.
- (3) This method is very effective at finding optimal or near optimal solution of MOLFP problems.

## VI. CONCLUSION

The proposed cooperative fuzzy game theoretic approach has been effectively applied to solve MOLFP, with no limitations in handling higher dimensional problems. The proposed algorithm keeps track of all the feasible solutions found during the optimization and therefore do not have any restrictions on the number of the Pareto-optimal solutions found. It is concluded that the inclusion of fuzzy membership function speeds-up the search process and helps in obtaining a fine-grained value for the objective functions. In this paper, we have addressed the problems of defining and characterizing solution concepts for a multi-objective linear fractional program problems. Owing to the simultaneous presence of the multiplicity of objective functions, we have privileged the satisfying approach rather than the optimizing one.

## REFERENCES

- [1] Aubin, J.P., "Cooperative fuzzy games", *Mathematics of Operations Research*, 6, 1– 13, 1981.
- [2] Aubin J., "Mathematical Methods of Game and Economic Theory", vol. 7, North-Holland, Amsterdam, The Netherlands, 1982.
- [3] Basar T, and Dyanamic G.J., "Noncooperative Game Theory", Academic Press, New York 1995.
- [4] Bellman R. E. and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, no. 4, pp. B141–B164, 1970.
- [5] Charnes, A., W.W. Cooper, "Programming with linear fractional functions," *Naval Res. Logist. Quart* 9, 181-186, 1962.
- [6] Chakraborty, M., and Sandipan Gupta, "Fuzzy mathematical programming for multi- Objective linear fractional programming problem", *Fuzzy Sets and System*. 125, 335- 342, 2002.
- [7] Dinkelbach W., "On Nonlinear Fractional Programming". *Management Science*, Vol. 13, 492-498, 1967.
- [8] Dhingra A. K. and Rao, S. S., "A cooperative fuzzy game theoretic approach to multiple objective design optimization," *European Journal of Operational Research*, vol. 83, no. 3, pp. 547–567, 1995.
- [9] Eren, Özceylan, Turan, Paksoy, "Fuzzy multi-objective linear programming approach for optimising a closed-loop supply chain network", *International Journal of Production Research*, Volume 51, Issue 8, pages 2443-2461, 2013.
- [10] Gilmore, P.C and Gomory, R.E., "A Linear Programming Approach to the Cutting Stock Problem-Part2", *Operational Research*, 11, 863-867, 1963.
- [11] Hwang, A.S.M. and Masud, A.S.M., "Multiple objective decision making-Method and Applications", *Lecture Notes in Economics and Mathematical system* 164, Springer- Verlag, Berlin, 1979.
- [12] Isbell, JR and Marlow, W.H., "Attraction Games", *Naval Research Logistics Quarterly*, 3, 71-93, 1956.
- [13] Ibrahim A. Baky., "Solving multi-level multi-objective linear programming problems through fuzzy goal programming approach", *Applied Mathematical Modelling*, Volume 34, Issue 9, pp. 2377–2387, 2010.
- [14] Ibrahim A. Baky., "Fuzzy goal programming algorithm for solving decentralized bi-level multi- objective programming problems". *Fuzzy Sets and Systems*, Volume 160, Issue 18, pp. 2701–2713, 2009.
- [15] Jagannathan, R., "On Some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming," *Management Science*, 12, 609- 615, 1966.
- [16] Kornbluth J.S.H, R.E Steur., "Goal programming with linear fractional criteria" *European Journal Operational Research*, 8, 58-65, 1981.
- [17] Klir, G.J and Yuan B., *Fuzzy Sets and Fuzzy Logic Theory and Applications*, Prentice Hall of India, 1997.
- [18] Lai Y.J., *Hierarchical Optimization: a satisfactory solution*, *Fuzzy Sets and Systems* 77, 321-335, 1996.
- [19] Lai, Y. J and Hwang C.L., *Fuzzy Mathematical Programming*, Springer-Verlag, Berlin, 1993.
- [20] Larbani, M., "Non cooperative fuzzy games in normal form: a survey," *Fuzzy Sets and Systems*, vol. 160, no. 22, pp. 3184–3210, 2009.
- [21] Liu B. (2002). *Theory and practice of uncertain programming*, Physical- Verlag, Heidelberg.
- [22] Luhandjula M.K., *Fuzzy Approaches for Multi Objective Linear Fractional Optimization*, *Fuzzy Sets and Systems*, 13, 11-23, 1984.
- [23] Martos, B., "Hyperbolic Programming", *Naval Research Logistics Quarterly*, 11, 135-155, 1964.
- [24] Nash J. F., "The bargaining problem," *Econometrica*, vol. 18, pp. 155–162, 1950.
- [25] Nash, J., "Two-person cooperative games," *Econometrica*, vol. 21, pp. 128–140, 1953.
- [26] Owen G., *Game Theory*, Academic Press New York. 1982.
- [27] Pan Y., M. J. Er, D. Huang, and Q. Wang, "Adaptive fuzzy control with guaranteed convergence of optimal approximation error," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 5, pp. 807–818, 2011.
- [28] Pan, Y. M. Huang, J. Er, D. and T. Sun, "Practical adaptive fuzzy  $H_\infty$  tracking control of uncertain nonlinear systems," *International Journal of Fuzzy Systems*, vol. 14, no. 4, pp. 463– 473, 2012.
- [29] Pan Y. P., M. J. Er, D. Huang, and Q. Wang, "Adaptive fuzzy control with guaranteed convergence of optimal approximation error," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 5, pp. 807–818, 2011.
- [30] Roy S.K., Biswal M.P. and Tiwari R.N., "Cooperative Fuzzy Game Theoretic Approach to Some Multi-Objective Linear Programming Problems", *The Journal of Fuzzy Mathematics*, Vol. 8, No. 3, pp. 635-643, 2000.
- [31] Sakawa M., Yumine T., *Interactive Fuzzy Decision Making for Multi Objective Linear Fractional Programming Problems. Large Scale Systems*, 5, 105-113, 1983.
- [32] Saaty T.L., *The Analytic Hierarchy Process: Planning, Priority Setting and Resource Allocation*, Rws Publication, Pttsburg, 1990.
- [33] Saaty T.L., *How to Make a Decision: The Analytic Hierarchy Process*, *European Journal of Operational Research*, 48 (1990), 9-26.
- [34] Stanly Lee E. and Li R.J. *Fuzzy multiple objective programming and compromise programming with Pareto optimum*, *Fuzzy Sets and Systems*. 53 (1993), 215-288.
- [35] Steur R.E., *Multiple Criteria Optimization: Theory, Computation and Application*, John Wiley and Sons, New York, 1986.
- [36] von Neumann J., Morgenstern O., *Theory of Games and Economic Behavior*, Wiley, New York, 1944.
- [37] Zadeh, L. A. "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [38] Zhihui Yang, Yizeng Chen et al., "Cooperative Fuzzy Games Approach to Setting Target Levels of ECs in Quality Function Deployment," *e Scientific World Journal*, Volume 2014, 9 pages, Article ID 673563.
- [39] Zimmermann H.J., *Fuzzy Set Theory and its Applications*, Kluwer-Nijhoff. Boston, MA, 1990.