



Conservation Laws of Electron Field in Standard Representation in Terms of Isotropic Complex Vectors

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Abstract: In previous works, using the standard representation for Dirac matrices, Dirac equation for half-spin particle has been written in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. Complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$ satisfy non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E} \cdot \vec{H} = 0$, obtained by equating to zero separately real and imaginary parts in equality $\vec{F}^2 = 0$.

In development of the above ideas, in this work we investigated the conservation laws of electron field in tensor formalism.

Keywords: Tensor formalism, standard representation, conservation laws, electron field.

I. Introduction

In previous works, using the standard representation for Dirac matrices, Dirac equation for half-spin particle has been written in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. It has been proved, that complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$ satisfy non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E} \cdot \vec{H} = 0$, obtained by equating to zero separately real and imaginary parts in equality $\vec{F}^2 = 0$. In addition, in those works, it has been proved that, vectors $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$ have the same properties as those of vectors $(\vec{E}, \vec{H})$, components of electromagnetic field. For example, under relativistic Lorentz transformations, they are transformed as components of a second rank tensor $F_{\mu\nu}$.

In development of the above ideas, in this work, we shall analyze the conservation laws of electron field in this formalism. We shall prove that, expressions for dynamical variables through vectors $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$ are the same as those obtained earlier by using the spinor representation for Dirac matrices.

II. Research Method

In this work, we shall investigate the conservation laws of energy, momentum, charge and spin for electron field in tensor formalism. Using Cartan map, spinor Dirac equation for half-spin particle has been written in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. Using the same method, the Lagrange function for electron field, written through spinors will be rewritten in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. With the use of Noether's theorem, we shall derive expressions for dynamical variables (energy, momentum, charge and spin) conserved in time.

III. Tensor Formulation of Dirac Equation in Standard Representation

Relativistic Dirac equation for half-spin particle with rest mass m has the form

$$(\gamma^\mu \partial_\mu - m)\Psi = 0. \quad (1)$$

We shall use the standard representation for γ^μ -matrices

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}. \quad (2)$$

Here $\vec{\sigma}$ are second rank Pauli spin matrices, having the form

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3)$$

In this representation, Dirac bispinor is written in the form

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \quad (4)$$

Where φ and χ are two components (but tridimensional) Pauli spinors.

Replacing formulas (2) and (4) in equation (1), we obtain a system of equations

$$\begin{cases} p_0 \phi - (\vec{p} \vec{\sigma}) \chi = -m \phi \\ p_0 \chi - (\vec{p} \vec{\sigma}) \phi = m \chi \end{cases} \quad (5)$$

Where $p_0 = i \frac{\partial}{\partial t}$
 $\vec{p} = -i \vec{\nabla}$.

$$(6)$$

Here we use the natural system of units in which $c=\hbar=1$.

In previous work, using Cartan map, equations (5) have been written in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$ as follows

$$\begin{cases} D_0 \vec{F} - \frac{i}{\sqrt{2}} \vec{D} (\vec{F} \vec{F}')^{1/2} + \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] = m \vec{F} \\ D_0 \vec{F}' - \frac{i}{\sqrt{2}} \vec{D} (\vec{F} \vec{F}')^{1/2} - \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] = -m \vec{F}' \end{cases} \quad (7)$$

IV. Conservation Laws of Electron Field in Standard Representation in Terms of Isotropic Vectors

Dirac equation (1) can be obtained by variation principle from Lagrange function

$$L = i (\bar{\Psi} \gamma^\mu \partial_\mu \Psi) - m \bar{\Psi} \Psi + \text{h.c.} \quad (8)$$

Using formulas (2) and (4), formula (8) can be represented in the form

$$L = \{\phi^+ [p_0 \phi - (\vec{p} \vec{\sigma}) \chi + m \phi] + \chi^+ [p_0 \chi - (\vec{p} \vec{\sigma}) \phi - m \chi]\}. \quad (9)$$

With the help of Cartan map, formula (9) can be written through isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$ as follows

$$L = \left\{ \left[D_0 \vec{F} - \frac{i}{\sqrt{2}} \vec{D} (\vec{F} \vec{F}')^{1/2} + \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] - m \vec{F} \right] \frac{\vec{F}^*}{(\vec{F} \vec{F}')^{1/2}} + \left[D_0 \vec{F}' - \frac{i}{\sqrt{2}} \vec{D} (\vec{F} \vec{F}')^{1/2} - \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] + m \vec{F}' \right] \frac{\vec{F}^*}{(\vec{F} \vec{F}')^{1/2}} \right\}. \quad (10)$$

Using Noether's theorem, we can derive from Lagrange function (10) expressions for fundamental dynamical variables (energy, momentum, charge and spin) conserved in time.

For energy we have

$$E = \int T^{00} dx^3. \quad (11)$$

Where

$$T^{00} = \frac{\partial L}{\partial \vec{F},0} \frac{\partial \vec{F}}{\partial x_0} + \frac{\partial L}{\partial \vec{F}',0} \frac{\partial \vec{F}'}{\partial x_0}. \quad (12)$$

Replacing formula (10) into formula (12), we find

$$T^{00} = i \frac{(\vec{F}^* \frac{\partial \vec{F}}{\partial t})}{2(\vec{F} \vec{F}')^{1/2}} + i \frac{(\vec{F}'^* \frac{\partial \vec{F}'}{\partial t})}{2(\vec{F} \vec{F}')^{1/2}}. \quad (13)$$

Considering expressions

$$\vec{F} = (\vec{E}^0 + i\vec{H}^0) \exp(-2i\varepsilon \mathcal{K}t + 2i\vec{k}\vec{r}), \quad (14)$$

$$\vec{F}' = (\vec{E}'^0 - i\vec{H}'^0) \exp(-2i\varepsilon \mathcal{K}t + 2i\vec{k}\vec{r}), \quad (15)$$

we obtain the following formula

$$T^{00} = \varepsilon \mathcal{K} (|\vec{E}| + |\vec{E}'|). \quad (16)$$

Here $\varepsilon = \pm 1$ is the sign of energy.

For momentum vector we have

$$p^j = \int T^{0j} dx^3, \quad (17)$$

where

$$T^{0j} = \frac{\partial L}{\partial \vec{F},0} \frac{\partial \vec{F}}{\partial x_j} + \frac{\partial L}{\partial \vec{F}',0} \frac{\partial \vec{F}'}{\partial x_j}. \quad (18)$$

Replacing formula (10) into formula (18) and using expressions (14)-(15), we obtain

$$T^{0j} = k_j (|\vec{E}| + |\vec{E}'|). \quad (19)$$

Similarly, for charge we have

$$Q = \int j^0 dx^3. \quad (20)$$

Here

$$j^0 = i \left[\left(\vec{F}^* \frac{\partial L}{\partial \vec{F}^*,0} - \frac{\partial L}{\partial \vec{F},0} \vec{F} \right) + \left(\vec{F}'^* \frac{\partial L}{\partial \vec{F}'^*,0} - \frac{\partial L}{\partial \vec{F}',0} \vec{F}' \right) \right]. \quad (21)$$

Replacing formula (10) into formula (21) and considering expressions (14)-(15), we find

$$j^0 = |\vec{E}| + |\vec{E}'|. \quad (22)$$

Finally, for spin pseudo vector, we find

$$\vec{S} = \frac{i}{2} \left[\frac{\vec{F} \times \vec{F}^*}{(\vec{F} \cdot \vec{F}^*/2)^{1/2}} + \frac{\vec{F}' \times \vec{F}'^*}{(\vec{F}' \cdot \vec{F}'^*/2)^{1/2}} \right]. \quad (23)$$

Considering expressions (14)-(15), we obtain

$$\vec{S} = \frac{\vec{E} \times \vec{H}}{|\vec{E}|} - \frac{\vec{E}' \times \vec{H}'}{|\vec{E}'|}. \quad (24)$$

We notice that the same expressions (16), (19), (22) and (24) have been obtained in previous works, by using the spinor representation for Dirac matrices.

V. Discussion and Conclusion

In this work, we investigated the laws of conservation of energy, momentum, charge and spin for electron field in tensor formalism. Using Cartan map, the Lagrange function for electron field written through spinors, has been rewritten in tensor form, through two complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. Applying Noether's theorem, we derived expressions for fundamental dynamical variables (energy, momentum, charge and spin) conserved in time. Those expressions are completely the same as those obtained in previous works, by using the spinor representation for Dirac matrices. Therefore, we proved, that expressions for dynamical variables through the strengths $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$ do not depend on the used representation for Dirac matrices, as expected.

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