



## Invariant Lagrangian for Electron Field in Terms of Divisors

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**Abstract:** In previous works, spinor Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations, through two isotropic complex vectors  $\vec{F} = \vec{E} + i\vec{H}$  and  $\vec{F}' = \vec{E}' - i\vec{H}'$ . It has been proved, that vectors  $\vec{E}$  and  $\vec{H}$  have the same properties as those of  $\vec{E}$  and  $\vec{H}$ , components of electromagnetic field.

In development of the above ideas, in this work, we wrote the Lagrange function for electron field in terms of divisors.

**Keywords:** Invariant Lagrangian, electron field, divisors.

### I. Introduction

In previous works, using different methods, spinor Dirac equation for half-spin particle such as electron has been written in tensor form, in the form of non-linear Maxwell's like equations, through two isotropic complex vectors  $\vec{F} = \vec{E} + i\vec{H}$  and  $\vec{F}' = \vec{E}' - i\vec{H}'$ , satisfying non-linear condition  $\vec{F}^2 = 0$ . The last condition is equivalent to two conditions for real quantities  $\vec{E}^2 - \vec{H}^2 = 0$  and  $\vec{E} \cdot \vec{H} = 0$ , obtained by equating to zero separately real and imaginary parts in equality  $\vec{F}^2 = 0$ . It has been proved, that vectors  $(\vec{E}, \vec{H})$  and  $(\vec{E}', \vec{H}')$  have the same properties as those of vectors  $\vec{E}$  and  $\vec{H}$ , components of the antisymmetric tensor  $F_{\mu\nu}$  of electromagnetic field. For example, under relativistic Lorentz transformations, they are transformed as components of a second rank tensor  $F_{\mu\nu}$ . In addition, it has been proved, that the solution of these non-linear equations for free particle as well fulfils Maxwell's equations for vacuum (with zero at the right side).

In development of the above ideas, in this work, we shall write the Lagrange function for electron field in tensor form, in terms of divisors.

### II. Research Method

In this work, we shall elaborate the Lagrange formalism for electron field in tensor formalism, through divisors. Via Cartan map, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two "electromagnetic fields"  $(\vec{E}, \vec{H})$  and  $(\vec{E}', \vec{H}')$ . Further, these non-linear equations have been written in covariant form, through divisors  $K_\mu, m_\mu, K'_\mu, m'_\mu$ . In this work, using the same method of tensor analysis, we shall write the Lagrange function for electron field through divisors  $K_\mu, m_\mu, K'_\mu, m'_\mu$ .

### III. Dirac Equation through Divisors

In previous works, Dirac equation for half-spin particle with rest mass  $m$

$$(\gamma_\mu \partial_\mu - m) \psi = 0, \quad (1)$$

has been written in tensor form, through two isotropic complex vectors  $\vec{F} = \vec{E} + i\vec{H}$  and  $\vec{F}' = \vec{E}' - i\vec{H}'$  as follows

$$\begin{cases} D_0 \vec{F} + v_i (\vec{D} F_i) - i \vec{D} \times \vec{F} = -\frac{m}{\sqrt{2}} \frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \\ D_0 \vec{F}' - v'_i (\vec{D} F'_i) + i \vec{D} \times \vec{F}' = -\frac{m}{\sqrt{2}} \frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \end{cases} \quad (2)$$

Where

$$\begin{aligned} D_0 &= \frac{i}{2} \frac{\partial}{\partial t}, \\ \vec{D} &= -\frac{i}{2} \vec{\nabla}, \\ \vec{V} &= \frac{\vec{E} \times \vec{H}}{\vec{E}^2}. \end{aligned} \quad (3)$$

Here we use the natural system of units in which  $c = \hbar = 1$ .

Separating real and imaginary parts in equations (2), we obtain a system of non-linear Maxwell's like equations for two electromagnetic fields  $(\vec{E}, \vec{H})$  and  $(\vec{E}', \vec{H}')$

$$\begin{cases} \text{rot } \vec{E} + \frac{\partial \vec{H}}{\partial t} = v_i (\vec{\nabla} H_i) + m_j \vec{a} \\ \text{rot } \vec{H} - \frac{\partial \vec{E}}{\partial t} = -v_i (\vec{\nabla} E_i) + m_j \vec{v} \\ \text{rot } \vec{E}' + \frac{\partial \vec{H}'}{\partial t} = -v'_i (\vec{\nabla} H'_i) - m_j \vec{a} \\ \text{rot } \vec{H}' - \frac{\partial \vec{E}'}{\partial t} = v'_i (\vec{\nabla} E'_i) + m_j \vec{v} \end{cases} \quad (4)$$

Here

$$\vec{j}_a = \sqrt{2} \frac{(\vec{E} \times \vec{E}' + \vec{H} \times \vec{H}') \cos \varphi / 2 + (\vec{H} \times \vec{E}' + \vec{H}' \times \vec{E}) \sin \varphi / 2}{[(\vec{E} \vec{E}')^2 + (\vec{H} \vec{H}')^2 + 2(\vec{E} \vec{E}')(\vec{H} \vec{H}') + (\vec{E} \vec{H}')^2 + (\vec{E}' \vec{H})^2 + 2(\vec{E} \vec{H}')(\vec{E}' \vec{H})]^{1/4}}, \quad (5)$$

$$\vec{j}_v = -\sqrt{2} \frac{(\vec{E} \times \vec{E}' + \vec{H} \times \vec{H}') \sin \varphi / 2 + (\vec{H} \times \vec{E}' + \vec{H}' \times \vec{E}) \cos \varphi / 2}{[(\vec{E} \vec{E}')^2 + (\vec{H} \vec{H}')^2 + 2(\vec{E} \vec{E}')(\vec{H} \vec{H}') + (\vec{E} \vec{H}')^2 + (\vec{E}' \vec{H})^2 + 2(\vec{E} \vec{H}')(\vec{E}' \vec{H})]^{1/4}}. \quad (6)$$

$$\varphi = \arctan \frac{\vec{E}' \vec{H} - \vec{E} \vec{H}'}{\vec{E} \vec{E}' + \vec{H} \vec{H}'}. \quad (7)$$

In particular, if  $\varphi=0$ , i.e.,  $\vec{E}/\vec{E}'$  and  $\vec{H}/\vec{H}'$ , we find a simple system

$$\begin{cases} \text{rot } \vec{E} + \frac{\partial \vec{H}}{\partial t} = v_i (\vec{\nabla} H_i) + \sqrt{2} m \frac{\vec{E} \times \vec{E}' + \vec{H} \times \vec{H}'}{(\vec{E} \vec{E}' + \vec{H} \vec{H}')^{1/2}} \\ \text{rot } \vec{H} - \frac{\partial \vec{E}}{\partial t} = -v_i (\vec{\nabla} E_i) + \sqrt{2} m \frac{\vec{H} \times \vec{E}' + \vec{H}' \times \vec{E}}{(\vec{E} \vec{E}' + \vec{H} \vec{H}')^{1/2}} \\ \text{rot } \vec{E}' + \frac{\partial \vec{H}'}{\partial t} = -v'_i (\vec{\nabla} H'_i) - \sqrt{2} m \frac{\vec{E} \times \vec{E}' + \vec{H} \times \vec{H}'}{(\vec{E} \vec{E}' + \vec{H} \vec{H}')^{1/2}} \\ \text{rot } \vec{H}' - \frac{\partial \vec{E}'}{\partial t} = v'_i (\vec{\nabla} E'_i) + \sqrt{2} m \frac{\vec{H} \times \vec{E}' + \vec{H}' \times \vec{E}}{(\vec{E} \vec{E}' + \vec{H} \vec{H}')^{1/2}} \end{cases} \quad (8)$$

Let us introduce bivector fields  $\mathbb{F}_{\mu\nu}$  and  $\mathbb{F}'_{\mu\nu}$  as follows

$$\mathbb{F}_{\mu\nu} = F_{\mu\nu} + i \tilde{F}_{\mu\nu}, \quad \mathbb{F}'_{\mu\nu} = F'_{\mu\nu} + i \tilde{F}'_{\mu\nu}. \quad (9)$$

Where

$$\mathbb{F}_{\mu\nu} = K_\mu \wedge m_\nu, \quad \mathbb{F}'_{\mu\nu} = K'_\mu \wedge m'_\nu. \quad (10)$$

Here the sign “ $\wedge$ ” means antisymmetrization over indices  $\mu$  and  $\nu$ .

In formula (10), four vectors  $K_\mu$  and  $m_\mu$  called divisors, have the form

$$\begin{aligned} K_\mu &= \left( |\vec{E}|, \frac{\vec{E} \times \vec{H}}{|\vec{E}|} \right), & m_\mu &= \left( 0, \frac{\vec{F}}{|\vec{E}|} \right), \\ K'_\mu &= \left( |\vec{E}'|, \frac{\vec{E}' \times \vec{H}'}{|\vec{E}'|} \right), & m'_\mu &= \left( 0, \frac{\vec{F}'}{|\vec{E}'|} \right). \end{aligned} \quad (11)$$

Equations (8) written through divisors take the form

$$\begin{cases} D_\nu \mathbb{F}^{\mu\nu} = K^\nu \partial^\mu m_\nu - \frac{m}{\sqrt{2}} (K_\nu + K'_\nu) \tilde{G}^{\mu\nu} \\ D_\nu \mathbb{F}'^{\mu\nu} = -K'^\nu \partial^\mu m'_\nu + \frac{m}{\sqrt{2}} (K_\nu + K'_\nu) \tilde{G}^{\mu\nu} \end{cases} \quad (12)$$

Here

$$\tilde{G}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\sigma\tau} G_{\sigma\tau}. \quad (13)$$

Where

$$G_{\mu\nu} = m_\mu \wedge m'_\nu. \quad (14)$$

#### IV. Lagrangian for Electron Field in Terms of Divisors

Spinor Dirac equation (1) can be obtained by variation principle from Lagrange function

$$L = i (\bar{\Psi} \gamma^\mu \partial_\mu \Psi) - m \bar{\Psi} \Psi + \text{h.c.} \quad (15)$$

Formula (15), written through isotropic complex vectors  $\vec{F} = \vec{E} + i\vec{H}$  and  $\vec{F}' = \vec{E}' + i\vec{H}'$  takes the form

$$\begin{aligned} L = i \left\{ \left[ D_0 \vec{F} + v_i (\vec{D} F_i) - i \vec{D} \times \vec{F} + \frac{m}{\sqrt{2}} \frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] \frac{\vec{F}^*}{(\vec{F} \vec{F}')^{1/2}} + \left[ D_0 \vec{F}' - v'_i (\vec{D} F'_i) + i \vec{D} \times \vec{F}' + \right. \right. \\ \left. \left. \frac{m}{\sqrt{2}} \frac{\vec{F} \times \vec{F}'}{(\vec{F} \vec{F}')^{1/2}} \right] \frac{\vec{F}'^*}{(\vec{F} \vec{F}')^{1/2}} \right\}. \end{aligned} \quad (16)$$

Formula (16) can be written through divisors as follows

$$L = i \left\{ \left[ D_\nu \mathbb{F}^{\mu\nu} - K^\nu \partial^\mu m_\nu + \frac{m}{\sqrt{2}} (K_\nu + K'_\nu) \tilde{G}^{\mu\nu} \right] m_\mu^* + \left[ D_\nu \mathbb{F}'^{\mu\nu} + K'^\nu \partial^\mu m'_\nu - \frac{m}{\sqrt{2}} (K_\nu + K'_\nu) \tilde{G}^{\mu\nu} \right] m_\mu'^* \right\}. \quad (17)$$

## V. Discussion and Conclusion

In this work, we investigated the possibility of elaboration of the Lagrange formalism for electron field in tensor formalism. With the help of Cartan map, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two electromagnetic fields  $(\vec{E}, \vec{H})$  and  $(\vec{E}', \vec{H}')$ . Further, these non-linear equations have been written in covariant form through divisors  $K_\mu, m_\mu, K'_\mu, m'_\mu$ . In this work, we elaborated the Lagrange formalism for electron field in tensor formalism, in terms of divisors. We found expression for the Lagrange function for electron field through divisors  $K_\mu, m_\mu, K'_\mu, m'_\mu$ . It is clear that, applying Noether's theorem, one can derive expressions for dynamical variables (energy, momentum, charge and spin) conserved in time. These expressions for dynamical variables have been obtained in previous works.

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