



On Ternary Quadratic Diophantine Equation $x^2 + 2y^2 = 4z$

M.A.Gopalan¹, V.Geetha²

¹Professor, Department of Mathematics, Shrimathi Indira Gandhi College, Trichy, Tamil Nadu, INDIA.

²Assistant Professor, Department of Mathematics, Cauvery College for Women, Trichy, Tamil Nadu, INDIA.

Abstract: The ternary quadratic diophantine equation $x^2 + 2y^2 = 4z$ is analyzed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

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Notations:

$t_{m,n}$ = Polygonal number of rank n with sides m

$ct_{m,n}$ = Centered Polygonal number of rank n with sides m

P_n = Pronic number

G_n = Gnomonic number

S_n = Star number

CH_n = Centered hexagonal number

I. INTRODUCTION

Diophantine equations is an interesting concept, as it can be seen from [1-2]. For an extensive review of various problems one may refer [3-14]. This communication concerns with yet another interesting ternary quadratic diophantine equation $x^2 + 2y^2 = 4z$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-zero integral solution is

$$x^2 + 2y^2 = 4z \quad (1)$$

To start with, it is seen that (1) is satisfied by the following triples:

$$(\pm 2k, \pm 2k, 3k^2) \text{ and } (10k, \pm 2k, 27k^2)$$

However, we have other patterns of solutions which are illustrated as follows.

PATTERN- I

Introducing the linear transformations

$$x = X + 2T, y = X - T, z = 3u^2 \quad (2)$$

in (1), it is written as

$$X^2 + 2T^2 = 4u^2 \quad (3)$$

The general solution of the above equation is given by

$$\left. \begin{aligned} T &= 2pq \\ X &= 2p^2 - q^2 \\ 2u &= 2p^2 + q^2 \end{aligned} \right\} \quad (4)$$

Since our aim is to find integral solution, substitute q as $2Q$ in (4).
The value of X, T , and u are

$$\left. \begin{aligned} X &= 2p^2 - 4Q^2 \\ T &= 4pQ \\ u &= p^2 + 2Q^2 \end{aligned} \right\} \quad (5)$$

Using (5) in (2), we get the corresponding non-zero distinct integer solutions of (1) are found to be

$$\begin{aligned} x &= x(p, Q) = 2p^2 - 4Q^2 + 8pQ \\ y &= y(p, Q) = 2(p^2 - 2Q^2 - 2pQ) \\ z &= z(p, Q) = 3(p^2 + 2Q^2)^2 \end{aligned}$$

PROPERTIES

- 1) $x(p, p+1) - y(p, p+1) = 24t_{3,p}$.
- 2) $2z(p, Q)$ is a nasty number.
- 3) $12 - [x(p, p-1) + 2y(p, p-1)]$ is a nasty number.
- 4) $x(1, -1) + y(1, -1) + z(1, -1) \equiv 0 \pmod{13}$.
- 5) $x(1, Q) + y(1, Q) + 4t_{6,Q} \equiv 0 \pmod{2^2}$.
- 6) $x(1, Q) - y(1, Q) - 6G_Q \equiv 0 \pmod{6}$.
- 7) $z(p, 1) - 144DF_p - t_{28,p} - 2P_p - 5G_p \equiv 0 \pmod{17}$.
- 8) $4P_p - x(p, 1) - y(p, 1) \equiv 0 \pmod{2^3}$.
- 9) $x(p-1, p) - CH_p + 3G_p \equiv 0 \pmod{2}$.
- 10) $2(P_{p-1} - G_p) - y(p-1, 1) \equiv 0 \pmod{6}$.

Remark:

Instead of (2) one may consider the following transformation

$$x = X + 2T, y = X - T, z = 9(4k^2 + 8k + 12)u^2$$

Following the procedure presented in Pattern I, the other choices of solutions to (1) are obtained.

PATTERN- II

Assume that

$$z = z(a, b) = a^2 + 2b^2 \quad (6)$$

where a, b are non zero distinct integers

Write 4 as

$$4 = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9} \quad (7)$$

Substitute (6) and (7) in (1), we get

$$x^2 + 2y^2 = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9} (a^2 + 2b^2)$$

By the method of factorization, the above equation is written as

$$(x + i\sqrt{2}y)(x - i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9} (a + i\sqrt{2}b)(a - i\sqrt{2}b)$$

On comparing the positive and negative factors, we get

$$(x + i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})}{3} (a + i\sqrt{2}b) \quad (8)$$

$$(x - i\sqrt{2}y) = \frac{(2 - i4\sqrt{2})}{3}(a - i\sqrt{2}b) \quad (9)$$

On comparing real and imaginary parts either in (8) or (9), we have

$$\left. \begin{aligned} x &= \frac{1}{3}[2a - 8b] \\ y &= \frac{1}{3}[4a + 2b] \end{aligned} \right\} \quad (10)$$

Since our aim is to be find an integral solutions,

Substituting $b = m, a = 3k + m - 3$ in (6) and (10)

The corresponding non-zero distinct integer solution of (1) are found to be

$$\begin{aligned} x &= x(k, m) = 2(k - m - 1) \\ y &= y(k, m) = 4k + 2m - 4 \\ z &= z(k, m) = (3k + m - 3)^2 + 2m^2 \end{aligned}$$

PROPERTIES

- 1) $x(1, -1) + y(1, m) + z(1, m) = 3t_{4,m}$.
- 2) $2x(k, m) - y(k, m) + z(1, m) + 3m = 3p_{m-1}$.
- 3) $x(k, k) + y(k, k) + z(k, k) - 6p_{k-1} \equiv 0 \pmod{3}$.
- 4) $x(k, k-1) + y(k, k-1) - G_k + 5 \equiv 0 \pmod{4}$.
- 5) $6[z(m, m) - 4S_m - 5]$ is a nasty number.
- 6) $x(k, 1)y(k, 1) - t_{18,k} + 8G_k + k \equiv 0 \pmod{10}$.
- 7) $CH_k + t_{14,k} - x(k^2, 1) - y(k^2, 1) - z(1, k-1) \equiv 0 \pmod{2}$.
- 8) $z(k-1, 1) - 2ct_{9,k} + 19G_k - k \equiv 0 \pmod{6}$.
- 9) $y(1, m+1) + z(1, m+1) - 2(ct_{3,m} - G_m) - m \equiv 0 \pmod{5}$.
- 10) $2P_{k-1} - G_k - x((k-1)^2, 1) \equiv 0 \pmod{3}$.

PATTERN-III

Equation (1) can be written as

$$x^2 + 2y^2 = 4z * 1 \quad (11)$$

Write 1 as

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} \quad (12)$$

Using (7) and (12) in equation (11), we get

$$x^2 + 2y^2 = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9} \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} (a^2 + 2b^2)$$

By method of factorization,

$$(x + i\sqrt{2}y)(x - i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})(2 - i4\sqrt{2})}{9} \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} (a + i\sqrt{2}b)(a - i\sqrt{2}b)$$

On comparing the positive and negative factors, we get

$$(x + i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})}{3} \frac{(1 + i2\sqrt{2})}{3} (a + i\sqrt{2}b) \quad (13)$$

$$(x - i\sqrt{2}y) = \frac{(2 - i4\sqrt{2})}{3} \frac{(1 - i2\sqrt{2})}{3} (a - i\sqrt{2}b) \quad (14)$$

Consider the equation (13)

$$(x + i\sqrt{2}y) = \frac{(2 + i4\sqrt{2})}{3} \frac{(1 + i2\sqrt{2})}{3} (a + i\sqrt{2}b)$$

On comparing the real and imaginary parts from the above equation, we get

$$\left. \begin{aligned} x &= \frac{-2}{9} [7a + 8b] \\ y &= \frac{2}{9} [4a - 7b] \end{aligned} \right\} \quad (15)$$

Since our aim is to find integral solutions, substitute $b = m, a = 9k + 4m - 9$

The corresponding non-zero distinct integer solutions of equation (1) are found to be

$$x = x(k, m) = -2[7k + 4m - 7]$$

$$y = y(k, m) = -2[4k + m - 4]$$

$$z = z(k, m) = (9k + 4m - 9)^2 + 2m^2$$

PROPERTIES

- 1) $2(x(-1, m) - y(-1, 4m))$ is nasty number.
- 2) $3z(1, m)$ is nasty number.
- 3) $x(k, k) + z(1, k) + G_k - 18P_{k-1} \equiv 0 \pmod{13}$.
- 4) $z(k, 1) + x(k, 1) - 2(t_{83, k} + ct_{25, k}) \equiv 0 \pmod{25}$.
- 5) $x(-2k, 1) + y(-2k, 1) + 2(t_{8, k} - G_k) - S_k \equiv 0 \pmod{13}$.
- 6) $x(k, 1)y(k, 1) + z(k, 1) - 8t_{29, k} - 6G_k - 5P_{k-1} + 2t_{19, k} \equiv 0 \pmod{86}$.
- 7) $z(k - 1, 1) - 8t + 94G_k - 18t_{11, k} + k \equiv 0 \pmod{104}$.
- 8) $x(k^2, k) + -y(k^2, k) - 6G_k + CH_k = 1$.
- 9) $y(k - 1, k^2) + 2ct_{2, k} + 3G_k \equiv 0 \pmod{7}$.

GENERATION OF SOLUTIONS:

Let (x_0, y_0, z_0) be any given non-zero distinct integral points on the given equation. It is observed that each of the following triple of integers also satisfies (1).

Triple 1: (x_n, y_n, z_n)

$$\begin{aligned} \text{where} \quad x_n &= 3^{n-1} [(1 + 2(-1)^n)x_0 + (2 - 2(-1)^n)y_0] \\ y_n &= 3^{n-1} [(1 - (-1)^n)x_0 + (2 + (-1)^n)y_0] \\ z_n &= 9^n z_0 \end{aligned}$$

Triple 2: (x_n, y_n, z_n)

where

$$\begin{aligned} x_n &= x_0 \\ y_n &= y_0 + 2n \\ z_n &= 2ny_0 + z_0 + 2n^2 \end{aligned}$$

Triple 3: (x_n, y_n, z_n)

where

$$\begin{aligned} x_n &= x_0 + 4n \\ y_n &= y_0 \\ z_n &= 2nx_0 + z_0 + 4n^2 \end{aligned}$$

III. CONCLUSION

It is worth to mention that instead of (12) one may consider the following representation for 1

$$1 = \frac{(1 + i12\sqrt{2})(1 - i12\sqrt{2})}{289}$$

To conclude one may research for other patterns of solutions and their corresponding properties.

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