



An Introduction to Differentiation and Integration of Rhotrices

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Abstract: Rhotrices can occur with functions as their elements. As a result of this, we need to understand how to differentiate and integrate a rhotrix with respect to an independent variable that is present in a function. This paper will introduce to us the fundamental operations of differentiation and integration of rhotrices.

Keywords: rhotrix; rhotrix differentiation; rhotrix integration; variable; anti-derivative.

I. Introduction

The concept of mathematical arrays that lies in some way between two-dimensional vectors and (2×2) -dimensional matrices and matrix-tertions and noitrets were discussed in [1], as a result of this Ajibade in [2] introduced an object which is in some way between (2×2) -dimensional

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and (3×3) -dimensional $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ matrices, and he called such an object a rhotrix. Algebra of

rhotrices were initially introduced in [2] by Ajibade. Let R and Q be two rhotrices such that

$$R = \left\langle \begin{array}{ccc} a & & \\ b & h(R) & d \\ e & & \end{array} \right\rangle \text{ and } Q = \left\langle \begin{array}{ccc} f & & \\ g & h(Q) & j \\ k & & \end{array} \right\rangle \quad (1)$$

Ajibade in [2] defined the addition of these two rhotrices R and Q as:

$$R + Q = \left\langle \begin{array}{ccc} a + f & & \\ b + g & h(R) + h(Q) & d + j \\ e + k & & \end{array} \right\rangle,$$

and their multiplication as:

$$R \circ Q = \left\langle \begin{array}{ccc} ah(Q) + fh(R) & & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ eh(Q) + kh(R) & & \end{array} \right\rangle$$

Another multiplication method for rhotrices called row-column multiplication was introduced by Sani in [3] in an effort to answer some questions raised by Ajibade. Using the rhotrices R and Q as defined in (1), Sani in [3] illustrated the row-column multiplication of rhotrices as:

$$R \circ Q = \left\langle \begin{array}{ccc} af + dg & & \\ bf + eg & h(R)h(Q) & aj + dk \\ bj + ek & & \end{array} \right\rangle$$

A generalization of the row-column multiplication method for n -dimensional rhotrices was given by Sani in [4].

That is: given n -dimensional rhotrices $R_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle$ and $Q_n = \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$ the multiplication of R_n and

Q_n is as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_2=1}^t (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_2=1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \quad t = (n+1)/2.$$

The method of converting a rhotrix to a special matrix called ‘coupled matrix’ was suggested by Sani in [5]. The system $R_n x = b$ for which R_n is an n -dimensional rhotrix, x the unknown n -dimensional rhotrix vector and b the right-hand-side rhotrix vector was introduced by Aminu in [6], in this article, Aminu discuss the necessary and sufficient condition for the solvability of systems of the form $R_n x = b$, if a system is solvable it was shown how a solution can be found. To the best of the author’s knowledge, no work has been done on integration and differentiation of rhotrices. So it is the primary aim of this paper, to introduce to us how to integrate and differentiate rhotrices.

II. Rhotrix Differentiation

If the elements of a rhotrix R_n are functions of a variable t , then the rhotrix is called a rhotrix function of t .

$$R_n = R_n(t) = \langle a_{ij}(t), c_{lk}(t) \rangle$$

The differential coefficients of R_n with respect to “ t ” is defined as

$$\frac{d}{dt} R_n(t) = \left\langle \frac{d}{dt} a_{ij}(t), \frac{d}{dt} c_{lk}(t) \right\rangle,$$

and its n th order derivative with respect to t is defined as

$$\frac{d^n}{dt^n} R_n(t) = \left\langle \frac{d^n}{dt^n} a_{ij}(t), \frac{d^n}{dt^n} c_{lk}(t) \right\rangle, \quad \text{for } n = 1, 2, 3, \dots$$

Therefore the elements of the differentiated rhotrix $\frac{d}{dt} R_n(t)$ are the derivatives of the corresponding elements of $R_n(t)$. That is

$$\frac{d}{dt} R_n(t) = \left\langle \begin{array}{ccccccccc} & & & & \frac{d}{dt} a_{11}(t) & & & & \\ & & & & \frac{d}{dt} a_{21}(t) & \frac{d}{dt} c_{11}(t) & \frac{d}{dt} a_{12}(t) & & \\ & \frac{d}{dt} a_{31}(t) & \frac{d}{dt} c_{21}(t) & \frac{d}{dt} a_{22}(t) & \frac{d}{dt} c_{12}(t) & \frac{d}{dt} a_{13}(t) & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \frac{d}{dt} a_{t1}(t) & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \frac{d}{dt} a_{tt}(t) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & \frac{d}{dt} a_{t-2}(t) & \frac{d}{dt} c_{t-1t-2}(t) & \frac{d}{dt} a_{t-1t-1}(t) & \frac{d}{dt} c_{t-2t-1}(t) & \frac{d}{dt} a_{t-2t}(t) & & & \\ & \frac{d}{dt} a_{t-1}(t) & \frac{d}{dt} c_{t-1t-1}(t) & \frac{d}{dt} a_{t-1t}(t) & & & & & \\ & & \frac{d}{dt} a_{tt}(t) & & & & & & \end{array} \right\rangle$$

where $t = \frac{n+1}{2}$.

Theorem 1² (Derivative of the sum of two rhotrices)

Let $R_n(t)$ and $P_n(t)$ be two rhotrices, each with differential elements. Then

$$\frac{d}{dt} [R_n(t) + P_n(t)] = \frac{d}{dt} R_n(t) + \frac{d}{dt} P_n(t).$$

Proof

It follows immediately from the definition of the sum of two rhotrices.

Theorem 2 (Derivative the product of two rhotrices)

Let $R_n(t)$ and $P_n(t)$ be two rhotrices which are conformable for multiplication and each with differential elements. Then

$$\frac{d}{dt} [R_n(t)P_n(t)] = R_n(t) \frac{d}{dt} P_n(t) + P_n(t) \frac{d}{dt} R_n(t)$$

Proof

It follows immediately from the definition of the rhotrix product of two rhotrices as defined in [2].

Example 1

Find $\frac{d}{dt}R_5(t)$, given that

$$R_5(t) = \left\langle \begin{array}{cccc} & & t^2 & \\ \sin t & 2t & -t^3 + t^2 & \\ t^3 & 2t^2 + t & \cos t & 2 \\ \sin t & t & t + 1 & \\ & 1 & & \end{array} e^{2t} \right\rangle$$

Solution

$$\begin{aligned} \frac{d}{dt}R_5(t) &= \left\langle \begin{array}{cccc} \frac{d}{dt}t^2 & & & \\ \frac{d}{dt}\sin t & \frac{d}{dt}2t & \frac{d}{dt}(-t^3 + t^2) & \\ \frac{d}{dt}t^3 & \frac{d}{dt}(2t^2 + t) & \frac{d}{dt}\cos t & \frac{d}{dt}2 \\ \frac{d}{dt}\sin t & \frac{d}{dt}t & \frac{d}{dt}(t + 1) & \\ \frac{d}{dt}1 & & & \end{array} e^{2t} \right\rangle \\ &= \left\langle \begin{array}{cccc} 2t & & & \\ \cos t & 2 & -3t^2 + 2t & \\ 3t^2 & 4t + 1 & -\sin t & 0 \\ \cos t & 1 & 1 & \\ 0 & & & \end{array} 2e^{2t} \right\rangle \end{aligned}$$

Theorem 3

Let $R_n(t)$ be a rhotrix whose entries are differentiable, that is $R_n(t)$ is a differentiable rhotrix. Then

$$\frac{d}{dt}R_n^{-1}(t) = -R_n^{-1}(t)\frac{d}{dt}R_n(t)R_n^{-1}(t)$$

Proof

By the definition of the inverse of a rhotrix, we have

$$R_n^{-1}(t)R_n(t) = I \quad (2)$$

Differentiating both sides of (2), we get

$$\begin{aligned} R_n^{-1}(t)\frac{d}{dt}R_n(t) + \frac{d}{dt}R_n^{-1}(t)R_n(t) &= 0 \\ \frac{d}{dt}R_n^{-1}(t)R_n(t) &= -R_n^{-1}(t)\frac{d}{dt}R_n(t) \\ \frac{d}{dt}R_n^{-1}(t) &= -R_n^{-1}(t)\frac{d}{dt}R_n(t)R_n^{-1}(t) \end{aligned}$$

III. Rhotrix Integration

Just as we can find the derivative of a rhotrix R_n with respect to an independent variable t , we can as well find the anti-derivative or the definite integral of R_n . The integral of $R_n(t)$, either definite or indefinite, is obtained by integrating each element of $R_n(t)$. Thus

$$\int_a^b R_n(t)dt = \left[\int_a^b \langle a_{ij}(t), c_{lk}(t) \rangle dt \right] = \left[\left\langle \int_a^b a_{ij}(t)dt, \int_a^b c_{lk}(t)dt \right\rangle \right]$$

and

$$\int R_n(t)dt = \left[\int \langle a_{ij}(t), c_{lk}(t) \rangle dt \right] = \left[\left\langle \int a_{ij}(t)dt, \int c_{lk}(t)dt \right\rangle \right]$$

Therefore, we have

$$\int R_n(t) = \left\langle \begin{array}{cccccc} & & & \int a_{11}(t)dt & & \\ & & & \int a_{21}(t)dt & \int c_{11}(t)dt & \int a_{12}(t)dt \\ & \int a_{31}(t)dt & \int c_{21}(t)dt & \int a_{22}(t)dt & \int c_{12}(t)dt & \int a_{13}(t)dt \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \int a_{t1}(t)dt & \dots & \dots & \dots & \dots & \dots & \int a_{1t}(t)dt \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & \int a_{t-2}(t)dt & \int c_{t-1t-2}(t)dt & \int a_{t-1t-1}(t)dt & \int c_{t-2t-1}(t)dt & \int a_{t-2t}(t)dt \\ & \int a_{t-1}(t)dt & \int c_{t-1t-1}(t)dt & \int a_{t-1t}(t)dt & & \\ & & \int a_{tt}(t)dt & & & \end{array} \right\rangle$$

Theorem 4

Let $R_n(t)$ and $P_n(t)$ be rhotrices. Then

$$\int [R_n(t) + P_n(t)] dt = \int R_n(t) dt + \int P_n(t) dt.$$

Proof

It follows immediately from the definition of the sum of two rhotrices.

Example 2

Find $\int R_5(t) dt$; where $R_5(t)$ is as defined above in example 1.

Solution

$$\begin{aligned} \int R_5(t) dt &= \int \left\langle \begin{array}{cccc} & t^2 & & \\ \sin t & 2t & -t^3 + t^2 & \\ t^3 & 2t^2 + t & \cos t & 2 \\ \sin t & t & t + 1 & \\ & 1 & & \end{array} e^{2t} \right\rangle dt \\ &= \left\langle \begin{array}{cccc} \int (t^2)dt & & & \\ \int (\sin t)dt & \int (2t)dt & \int (-t^3 + t^2)dt & \\ \int (t^3)dt & \int (2t^2 + t)dt & \int (\cos t)dt & \int (2)dt & \int (e^{2t})dt \\ \int (\sin t)dt & \int (t)dt & \int (t + 1)dt & & \\ & \int (1)dt & & & \end{array} \right\rangle \\ &= \left\langle \begin{array}{cccc} \frac{t^3}{3} + c_1 & & & \\ -\cos t + c_6 & t^2 + c_4 & \frac{-t^4}{4} + \frac{t^3}{3} + c_2 & \\ \frac{t^4}{4} + c_{11} & \frac{2t^3}{3} + \frac{t^2}{2} + c_9 & \sin t + c_7 & 2t + c_5 & \frac{e^{2t}}{2} + c_3 \\ -\cos t + c_{12} & \frac{t^2}{2} + c_{10} & \frac{t^2}{2} + t + c_8 & & \\ & t + c_{13} & & & \end{array} \right\rangle \end{aligned}$$

Theorem 5

For any rhotrix $R_n(t)$,

$$\text{tr} \left(\int R_n(t) dt \right) = \int \text{tr} (R_n(t)) dt$$

Proof

$$\text{tr} \left(\int R_n(t) dt \right) = \sum_{i=1}^t \sum_{l=1}^{t-1} \left(\int \langle a_{ij}(t), c_{lk}(t) \rangle dt \right)_{ii,ll}$$

$$\begin{aligned}
 &= \sum_{i=1}^t \sum_{l=1}^{t-1} \left(\int \left\langle \sum_{j=1}^t a_{ij}(t), \sum_{k=1}^{t-1} c_{lk}(t) \right\rangle dt \right) \\
 &= \int \left\langle \sum_{i=1}^t \sum_{j=1}^t a_{ij}(t), \sum_{l=1}^{t-1} \sum_{k=1}^{t-1} c_{lk}(t) \right\rangle dt \\
 &= \int \left\langle \sum_{i=1}^t a_{ii}(t), \sum_{l=1}^{t-1} c_{ll}(t) \right\rangle dt \\
 &= \int \text{tr}(R_n(t)) dt
 \end{aligned}$$

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