



New Classes Containing Combination of Ruscheweyh Derivative and a New Generalized Multiplier Differential Operator

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Abstract: New classes containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier differential operator have been introduced. Sharp results concerning coefficients and distortion theorems of functions belonging to these classes are determined.

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I. Introduction

Denote by U the open unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U . Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

The author has recently introduced the following new generalized multiplier differential operator in [15].

Definition 1.1 Let $m \in N_0 = N \cup \{0\}$, $\beta \geq 0$, α a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I_{\alpha, \beta}^m$ was defined by

$$I_{\alpha, \beta}^0 f(z) = f(z), I_{\alpha, \beta}^1 f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta}, \dots, I_{\alpha, \beta}^m f(z) = I_{\alpha, \beta}(I_{\alpha, \beta}^{m-1} f(z)).$$

Remark 1.2 Observe that for $f(z)$ given by (1.1), we have

$$I_{\alpha, \beta}^m f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha, \beta, m) a_k z^k, \quad (1.2)$$

where

$$A_k(\alpha, \beta, m) = \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m. \quad (1.3)$$

We note that: i) $I_{1-\beta, \beta, 0}^m f(z) = D_{\beta}^m f(z)$, $\beta \geq 0$ (See F. M. Al-Oboudi [1]), ii) $I_{l+1-\beta, \beta, 0}^m f(z) = I_{l, \beta}^m f(z)$, $l > -1$, $\beta \geq 0$ (See A. Catas [3] and he has considered for $l \geq 0$) and iii) $I_{\alpha, 1}^m f(z) = I_{\alpha}^m f(z)$, $\alpha > -1$ (See Cho and Srivastava [4]) and Cho and Kim [5]). $D_1^m f(z)$ was introduced by Salagean [9] and was considered for $m \geq 0$ in [2].

Definition 1.3 ([8]) For $m \in N_0$, $f \in A$, the operator R^m is defined by $R^m : A \rightarrow A$, $R^0 f(z) = f(z)$, $R^1 f(z) = z f'(z)$, $\dots$, $(m+1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z)$, $z \in U$.

Remark 1.4 If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then $R^m f(z) = z + \sum_{k=2}^{\infty} B_k(m) a_k z^k$, $z \in U$, where

$$B_k(m) = \frac{(m+k-1)!}{m!(k-1)!}. \quad (1.4)$$

The author has introduced the following operator in [16].

Definition 1.5 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.

Denote by $RI_{\alpha, \beta, \delta}^m$, the operator given by $RI_{\alpha, \beta, \delta}^m : A \rightarrow A$,

$$RI_{\alpha, \beta, \delta}^m f(z) = (1-\delta)R^m f(z) + \delta I_{\alpha, \beta}^m f(z), z \in U.$$

The operator was studied also in [11], [12], [13] and [14]. Clearly $RI_{\alpha, \beta, 0}^m = R^m$ and $RI_{\alpha, \beta, 1}^m = I_{\alpha, \beta}^m$.

Remark 1.6 If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then from (1.2) and Remark 1.4, we have

$$RI_{\alpha, \beta, \delta}^m f(z) = z + \sum_{k=2}^{\infty} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k, z \in U,$$

where $A_k(\alpha, \beta, m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively.

We introduce new classes as below by making use of the generalized operator $RI_{\alpha, \beta, \delta}^m$.

Definition 1.7 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in (0,1], \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathfrak{I}_{\alpha, \beta, \delta}^m(\sigma, \rho)$ if and only if

$$\left| \frac{\frac{[z^2(RI_{\alpha, \beta, \delta}^m f(z))']}{2(RI_{\alpha, \beta, \delta}^m f(z))} - 1}{\frac{[z^2(RI_{\alpha, \beta, \delta}^m f(z))']}{2(RI_{\alpha, \beta, \delta}^m f(z))} + 1 - 2\rho} \right| < \sigma, z \in U. \quad (1.5)$$

Definition 1.8 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in (0,1], \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\ell_{\alpha, \beta, \delta}^m(\sigma, \rho)$ if and only if

$$\left| \frac{\frac{2z[z(RI_{\alpha, \beta, \delta}^m f(z))']}{[z(RI_{\alpha, \beta, \delta}^m f(z))']} - 1}{\frac{2z[z(RI_{\alpha, \beta, \delta}^m f(z))']}{[z(RI_{\alpha, \beta, \delta}^m f(z))']} + 1 - 2\rho} \right| < \sigma, z \in U. \quad (1.6)$$

Definition 1.9 Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1], \sigma \in (0,1], \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathfrak{R}_{\alpha, \beta, \delta}^m(\sigma, \rho)$ if and only if

$$\left| \frac{\left(\frac{z(RI_{\alpha, \beta, \delta}^m f(z))'}{RI_{\alpha, \beta, \delta}^m f(z)} \right) \left(1 + \frac{z(RI_{\alpha, \beta, \delta}^m f(z))''}{(RI_{\alpha, \beta, \delta}^m f(z))'} \right) - 1}{\left(\frac{z(RI_{\alpha, \beta, \delta}^m f(z))'}{RI_{\alpha, \beta, \delta}^m f(z)} \right) \left(1 + \frac{z(RI_{\alpha, \beta, \delta}^m f(z))''}{(RI_{\alpha, \beta, \delta}^m f(z))'} \right) + 1 - 2\rho} \right| < \sigma, z \in U. \quad (1.7)$$

Let T denote the subclass of A consisting of functions whose non-zero coefficients, from second on, are negative; that is, an analytic function f is in T if and only if it can be expressed as $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$,

$a_k \geq 0, z \in U$. If $f \in T$, then $RI_{\alpha, \beta, \delta}^m f(z) = z - \sum_{k=2}^{\infty} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k$, where

$A_k(\alpha, \beta, m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively. We denote by $T\mathfrak{I}_{\alpha, \beta, \delta}^m(\sigma, \rho)$, $T\ell_{\alpha, \beta, \delta}^m(\sigma, \rho)$ and $T\mathfrak{R}_{\alpha, \beta, \delta}^m(\sigma, \rho)$, the classes of functions $f(z) \in T$ satisfying (1.5), (1.6) and (1.7) respectively.

In this paper, sharp results concerning coefficients and distortion theorems for the classes $T\mathfrak{I}_{\alpha, \beta, \delta}^m(\sigma, \rho)$, $T\ell_{\alpha, \beta, \delta}^m(\sigma, \rho)$ and $T\mathfrak{R}_{\alpha, \beta, \delta}^m(\sigma, \rho)$ are determined. Throughout this paper, unless otherwise mentioned we shall assume that $A_k(\alpha, \beta, m)$ and $B_k(m)$ are as defined in (1.3) and (1.4) respectively.

II. Coefficient bounds.

In this section we study the characterization properties following the papers of V. P. Gupta and P. K. Jain [6, 7] and H. Silverman [10].

Theorem 2.1 A function f is in $T\mathfrak{S}_{\alpha,\beta,\delta}^m(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} \{k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2\} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k \leq 4\sigma(1-\rho). \quad (2.1)$$

The result is sharp.

Proof. Suppose f satisfies (2.1). Then for $|z| < 1$, we have

$$\begin{aligned} & \left| [z^2 (RI_{\alpha,\beta,\delta}^m f(z))]' - 2RI_{\alpha,\beta,\delta}^m f(z) \right| - \sigma \left| [z^2 (RI_{\alpha,\beta,\delta}^m f(z))]' + 2(1-2\rho)RI_{\alpha,\beta,\delta}^m f(z) \right| = \\ & \left| - \sum_{k=2}^{\infty} (k(k+1) - 2) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k \right| - \\ & \sigma \left| 4(1-\rho) - \sum_{k=2}^{\infty} (k(k+1) + 2(1-2\rho)) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k \right| \leq \\ & \sum_{k=2}^{\infty} (k(k+1) - 2) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k - 4\sigma(1-\rho) + \\ & \sum_{k=2}^{\infty} \sigma(k(k+1) + 2(1-2\rho)) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k = \\ & \sum_{k=2}^{\infty} [k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2] \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k - 4\sigma(1-\rho) < 0. \end{aligned}$$

Hence, by using the maximum modulus theorem and (1.5), $f \in T\mathfrak{S}_{\alpha,\beta,\delta}^m(\sigma, \rho)$.

For the converse, assume that

$$\left| \frac{\frac{[z^2 (RI_{\alpha,\beta,\delta}^m f(z))]' - 1}{2(RI_{\alpha,\beta,\delta}^m f(z))} - 1}{\frac{[z^2 (RI_{\alpha,\beta,\delta}^m f(z))]' + 1 - 2\rho}{2(RI_{\alpha,\beta,\delta}^m f(z))}} \right| = \left| \frac{- \sum_{k=2}^{\infty} (k(k+1) - 2) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k}{4\sigma(1-\rho) - \sum_{k=2}^{\infty} \sigma(k(k+1) + 2(1-2\rho)) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k} \right| < \sigma, z \in U.$$

Since $\operatorname{Re}(z) \leq |z|$ for all $z \in U$, we obtain

$$\operatorname{Re} \left(\frac{\sum_{k=2}^{\infty} (k(k+1) - 2) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k}{4\sigma(1-\rho) - \sum_{k=2}^{\infty} \sigma(k(k+1) + 2(1-2\rho)) \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k} \right) < \sigma. \quad (2.2)$$

Choose values of z on the real axis so that $([z^2 (RI_{\alpha,\beta,\delta}^m f(z))]' / 2RI_{\alpha,\beta,\delta}^m f(z))$ is real. Upon clearing the denominator in (2.2) and letting $z \rightarrow 1$ through real values, we have the desired inequality (2.1).

The function $f_1(z) = z - \frac{4\sigma(1-\rho)}{\{k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2\} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}} z^k$,

$k \geq 2$, is an extremal function for the theorem.

Theorem 2.2 A function f is in $T\ell_{\alpha,\beta,\delta}^m(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} \{2k^2(1+\sigma) + (k+1)(\sigma(1-2\rho) - 1)\} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k \leq 4\sigma(1-\rho). \quad (2.3)$$

The result is sharp.

Theorem 2.3 A function f is in $T\mathfrak{R}_{\alpha,\beta,\delta}^m(\sigma, \rho)$ if and only if

$$\sum_{k=2}^{\infty} \{k^2(1+\sigma) + \sigma(1-2\rho) - 1\} \{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k \leq 2\sigma(1-\rho). \quad (2.4)$$

The result is sharp.

The proofs of Theorem 2.2 and Theorem 2.3 are similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

$$f_2(z) = z - \frac{4\sigma(1-\rho)}{\{2k^2(1+\sigma) + (k+1)(\sigma(1-2\rho) - 1)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}} z^k, k \geq 2$$

and

$$f_3(z) = z - \frac{2\sigma(1-\rho)}{\{(k^2(1+\sigma) + \sigma(1-2\rho) - 1)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}}, k \geq 2,$$

respectively.

Corollary 2.4

i) If $f \in T\mathfrak{S}_{\alpha, \beta, \delta}^m(\sigma, \rho)$

then $a_k \leq \frac{4\sigma(1-\rho)}{\{k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}}, k \geq 2$, with equality only for the functions of the form $f_1(z)$.

ii) If $f \in T\ell_{\alpha, \beta, \delta}^m(\sigma, \rho)$

then $a_k \leq \frac{4(1-\rho)}{\{(2k^2(1+\sigma) + (k+1)(\sigma(1-2\rho) - 1)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}}, k \geq 2$, with equality only for the functions of the form $f_2(z)$.

iii) If $f \in T\mathfrak{R}_{\alpha, \beta, \delta}^m(\sigma, \rho)$

then $a_k \leq \frac{2\sigma(1-\rho)}{\{(k^2(1+\sigma) + \sigma(1-2\rho) - 1)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\}}, k \geq 2$, with equality only for the functions of the form $f_3(z)$.

III. Distortion theorems

Theorem 3.1 If a function $f(z) \in T$ is in $T\mathfrak{S}_{\alpha, \beta, \delta}^m(\sigma, \rho)$ then

$$|f(z)| \geq |z| - \frac{\sigma(1-\rho)}{(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{\sigma(1-\rho)}{(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U.$$

Proof In view of Theorem 2.1, we have

$$4(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\} \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \{k(k+1)(1+\sigma) + 2\sigma(1-2\rho) - 2\}\{(1-\delta)B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k \leq 4\sigma(1-\rho).$$

Thus $\sum_{k=2}^{\infty} a_k \leq \frac{\sigma(1-\rho)}{(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, \delta)\}}$. So we get

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{\sigma(1-\rho)}{(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2.$$

On the other hand

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{\sigma(1-\rho)}{(1+\sigma(2-\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2.$$

Theorem 3.2 If a function $f(z) \in T$ is in $T_{\alpha, \beta, \delta}^{\ell^m}(\sigma, \rho)$ then

$$|f(z)| \geq |z| - \frac{4\sigma(1-\rho)}{(5+\sigma(11-6\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{4\sigma(1-\rho)}{(5+\sigma(11-6\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U.$$

Theorem 3.3 If a function $f(z) \in T$ is in $T_{\alpha, \beta, \delta}^{\mathcal{R}^m}(\sigma, \rho)$ then

$$|f(z)| \geq |z| - \frac{2\sigma(1-\rho)}{(3+\sigma(5-2\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{2\sigma(1-\rho)}{(3+\sigma(5-2\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} |z|^2, z \in U.$$

The proofs of Theorem 3.2 and Theorem 3.3 are similar to that of Theorem 3.1.

Remark 3.4 The bounds of Theorem 3.2 and Theorem 3.3 are sharp since the equalities are attained for the

functions $f(z) = z - \frac{4\sigma(1-\rho)}{(5+\sigma(11-6\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} z^2$ ($z = \pm r$) and

$f(z) = z - \frac{2\sigma(1-\rho)}{(3+\sigma(5-2\rho))\{(m+1)(1-\delta) + \delta A_2(\alpha, \beta, m)\}} z^2$ ($z = \pm r$), respectively

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