



Lagrange Formalism for Electromagnetic Field in Terms of Divisors

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Abstract: In previous works, with the help of Cartan map, Weyl's equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. It has been proved, that the complex vector $\vec{F} = \vec{E} + i\vec{H}$ satisfies non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E} \cdot \vec{H} = 0$, obtained by separating real and imaginary parts in equality $\vec{F}^2 = 0$. Further, it has been proved, that the vectors $\vec{E}$ and $\vec{H}$ have the same properties as vectors $\vec{E}$ and $\vec{H}$, components of electromagnetic field. For example, under Lorentz relativistic transformations, they are transformed as components of a second rank tensor $F_{\mu\nu}$. In the works that followed, Weyl's equation has been written in tensor form, through divisors K_μ and m_μ .

In this work, by analogy with Weyl's equation, we wrote Maxwell's equations through divisors K_μ and m_μ , and we elaborated the Lagrange formalism for electromagnetic field in terms of divisors K_μ and m_μ .

Keywords: Lagrange formalism, electromagnetic field, divisors.

I. Introduction

The Lagrange formalism is one of the main tools of the study of the dynamic systems of a vast variety of physical systems, including systems with finite and infinite number of degrees of freedom.

In previous works, via Cartan map, Weyl's equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. It has been proved, that the vectors $\vec{E}$ and $\vec{H}$ have the same properties as vectors $\vec{E}$ and $\vec{H}$, components of electromagnetic tensor $F_{\mu\nu}$. For example, under Lorentz relativistic transformations, they are transformed as components of a second rank tensor $F_{\mu\nu}$. Further, it has been proved, that the solution of Weyl's equation in tensor formalism for free particle as well fulfils Maxwell's equations for vacuum (with zero at the right side). In the development of the above ideas, in the works that followed, Weyl's equation has been written in covariant form, through divisors K_μ and m_μ . By analogy with Weyl's equation, Maxwell's equations also have been written through divisors K_μ and m_μ .

In this work, we shall elaborate Lagrange formalism for electromagnetic field in terms of divisors K_μ and m_μ .

II. Research Method

In previous works, Weyl's equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. In the development of these ideas, Weyl's equation for neutrino has been written through divisors K_μ and m_μ . In this work, using the same mathematical method of tensor analysis, we shall write Maxwell's equations through divisors K_μ and m_μ and we shall elaborate the Lagrange formalism for electromagnetic field in terms of divisors K_μ and m_μ .

A. Maxwell's Equations in Terms of Divisors

In previous works, Weyl's equation for neutrinos

$$p_0 \xi = (\vec{p} \vec{\sigma}) \xi, \quad (1)$$

has been written in tensor form as follows

$$D_0 \vec{F} = i \vec{D} \times \vec{F} - (\vec{D} F_i) v_i, \quad (2)$$

where $\vec{F} = \vec{E} + i\vec{H}$.

Here

$$\begin{aligned} D_0 &= i \frac{\hbar}{2} \frac{\partial}{\partial t}, \\ \vec{D} &= -i \frac{\hbar}{2} \vec{\nabla}, \\ \vec{v} &= \frac{\vec{E} \times \vec{H}}{E^2}. \end{aligned} \quad (3)$$

For simplicity, in the following we shall use the natural system of units in which $c=\hbar=1$.

Separating real and imaginary parts in equation (2), we obtain a system of non-linear Maxwell's like equations

$$\begin{cases} \text{rot } \vec{E} + \frac{\partial \vec{H}}{\partial t} = v_i (\vec{\nabla} H_i) \\ \text{rot } \vec{H} - \frac{\partial \vec{E}}{\partial t} = -v_i (\vec{\nabla} E_i) \end{cases} \quad (4)$$

In terms of isotropic vectors, Maxwell's equations for vacuum take the form

$$\begin{cases} D_0 \vec{F} = i \vec{D} \times \vec{F} \\ \vec{D} \vec{F} = 0 \end{cases} \quad (5)$$

However, in general the solution of Maxwell's equations does not satisfy non-linear isotropic condition $\vec{F}^2 = 0$, whereas the solution of neutrino equation (2) always satisfies this condition.

Let us introduce bivector field

$$F_{\mu\nu} = K_\mu \wedge m_\nu, \quad (6)$$

satisfying isotropic condition

$$F_{\mu\nu} \tilde{F}_{\mu\nu} = 0. \quad (7)$$

Here K_μ is a real four vector; m_μ is a complex four vector. In formula (7), the sign " $\wedge$ ", means antisymmetrization over indices μ and ν .

The four vectors K_μ and m_μ are called divisors and are expressed through $\vec{E}$ and $\vec{H}$ as follows

$$K_\mu = \left(|\vec{E}|, \frac{\vec{E} \times \vec{H}}{E^2} \right), \quad m_\mu = \left(0, \frac{\vec{F}}{|\vec{E}|} \right). \quad (8)$$

Equations (4) written through divisors take the form

$$D^\nu F_{\mu\nu} = K^\nu \partial_\mu m_\nu. \quad (9)$$

Maxwell's equations (5) written through divisors take the form

$$D^\nu F_{\mu\nu} = 0. \quad (10)$$

B. Lagrangian of Electromagnetic Field in Terms of Divisors

In spinor formalism, Weyl's equation for neutrinos (1) can be obtained by variation principle from Lagrange function

$$L = \frac{i}{2} (\xi \sigma^\mu \partial_\mu \xi^* - \partial^\mu \xi \sigma_\mu \xi^*). \quad (11)$$

Written through isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$, formula (11) takes the form

$$L = \frac{i}{2} \{ [D_0 \vec{F} - i \vec{D} \times \vec{F} + v_i (\vec{D} F_i)] \vec{F}^* - [D_0 \vec{F}^* + i \vec{D} \times \vec{F}^* + v_i (\vec{D} F_i^*)] \vec{F} \} / \left(\frac{\vec{F} \vec{F}^*}{2} \right)^{1/2}. \quad (12)$$

From Lagrange function (12), we immediately obtain the Lagrange function for electromagnetic field in the form

$$L = \frac{i}{2} \{ [D_0 \vec{F} - i \vec{D} \times \vec{F}] \vec{F}^* - [D_0 \vec{F}^* + i \vec{D} \times \vec{F}^*] \vec{F} \} / \left(\frac{\vec{F} \vec{F}^*}{2} \right)^{1/2}. \quad (13)$$

When calculating the variations, the quantity $\left(\frac{\vec{F} \vec{F}^*}{2} \right)^{1/2}$ is considered as a constant.

It is easy to prove that, the Lagrange function (13), written through divisors takes the form

$$L = (D^\nu F_{\mu\nu}) m^{\mu*} - (D^\nu F_{\mu\nu}^*) m^\mu. \quad (14)$$

C. Fundamental Dynamical Variables

Using Noether's theorem, we can derive from lagrangian (14) expressions for fundamental dynamical physical quantities.

Energy is determined by the formula

$$E = \int T^{00} d^3x, \quad (15)$$

where

$$T^{00} = \frac{\partial L}{\partial m_{\nu,0}} m_{\nu,0} + \frac{\partial L}{\partial m_{\nu,0}^*} m_{\nu,0}^*. \quad (16)$$

Replacing expression (14) in formula (16), we find

$$T^{00} = \frac{i}{4} \left(m^{\nu*} \frac{\partial m_\nu}{\partial t} - m^\nu \frac{\partial m_{\nu}^*}{\partial t} \right). \quad (17)$$

Considering the solution for free particle in the form of plane waves

$$m_\nu = \left[0, \frac{(\vec{E}^0 + i\vec{H}^0)}{|\vec{E}|} e^{-2ikt + 2i\vec{k}\vec{r}} \right], \quad (18)$$

we find

$$T^{00} = k |\vec{E}|. \quad (19)$$

Similarly for momentum, we have

$$P^j = \int T^{0j} d^3x, \quad (20)$$

where

$$T^{0j} = \frac{i}{4} [m^v \vec{\nabla}_j m_v^* - m^{v*} \vec{\nabla}_j m_v]. \quad (21)$$

Replacing expression (18) in formula (21), we obtain

$$T^{0j} = k_j |\vec{E}|. \quad (22)$$

For charge, we find

$$Q = \int j^0 d^3x, \quad (23)$$

where

$$j^0 = i \left(m_v^* \frac{\partial L}{\partial m_{v,0}^*} - m_v \frac{\partial L}{\partial m_{v,0}} \right). \quad (24)$$

Using expression (14), we obtain

$$j^0 = m^v m_v^* + m_v^* m^v. \quad (25)$$

Replacing expression (18) in formula (25), we get

$$j^0 = |\vec{E}|. \quad (26)$$

In the same way, the density of the spin pseudo vector is determined by the formula

$$s_k = \frac{1}{2} \epsilon_{klm} s_{lm}^0, \quad (27)$$

where

$$s_{lm}^0 = \left[-\frac{\partial L}{\partial m_{i,0}} m_j A_{i,lm}^j - \frac{\partial L}{\partial m_{i,0}^*} m_j^* A_{i,lm}^j \right]. \quad (28)$$

Here

$$A_{i,lm}^j = g_{il} \delta_m^j - g_{im} \delta_l^j. \quad (29)$$

Replacing expressions (14) in formula (28), we obtain

$$\vec{s} = \frac{\vec{E} \times \vec{H}}{|\vec{E}|}. \quad (30)$$

III. Discussion and Conclusion

In this work, we investigated Maxwell's equations. It is known that, Maxwell's equations can be written in different forms. Here we proved that, they can also be written through divisors. Similarly with neutrinos field, we obtained the Lagrange function for electromagnetic field in terms of divisors. Using Noether's theorem we derived expressions for fundamental dynamical variables (energy, momentum, charge and spin) conserved in time.

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