



Some Properties of a Subclass of Univalent Functions Related to Hypergeometric Function and its Integral

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Abstract: In this paper we defined class $S(\alpha, \beta, \lambda)$ by using the class $S(\lambda, \alpha)$ (see[1]) and obtained condition for hypergeometric function $F(a,b;c;z)$ and its integral to be in this class.

Keywords: univalent functions, starlike functions, convex functions, differential operator, hypergeometric function.

I. Introduction

Let A denotes the class of functions of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. Let S be subclass of A , of functions univalent in U and T be subclass of S , consisting of functions of the form,

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0) \quad (1.2)$$

Goodman [2] introduced classes UCV and UST of uniformly convex and uniformly starlike functions. Further Ronning studied classes UCV and UST (for more details see [3]). Among several interesting definitions given in literature (See, [4], [5], [6], [7] and many others) we recall the following,

Definition 1.1

The function $f(z)$ defined by Equation (1.1), which satisfies the condition,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|$$

where, $0 \leq \alpha < 1$, $\beta \geq 0$ is called as β - Uniformly starlike function of order α .

The class of β - Uniformly starlike function of order α is denoted by $\beta - S(\alpha)$.

- When $\beta = 0$, i.e. if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha$, the function $f(z)$ is called starlike of order α .
- When $\beta = 1$, the function $f(z) \in UST$
- The function $f(z)$ is called starlike, if both α and β are zero.

Definition 1.2

The function $f(z)$ defined by Equation (1.1), which satisfies the condition,

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| \frac{zf''(z)}{f'(z)} \right|$$

where, $-1 \leq \alpha < 1$, $\beta \geq 0$ is called β - Uniformly Convex function of order α .

The class of β - Uniformly convex function of order α is denoted by $\beta - K(\alpha)$.

- When $\beta = 0$, i.e. $\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha$, if the function $f(z)$ is called convex of order α .
- When $\beta = 1$, the function $f(z) \in UCV$
- The function $f(z)$ is called convex, if both α and β are zero.

The classes $\beta - S(\alpha)$ and $\beta - K(\alpha)$ were studied by Goodman, Ronning and Minda and Ma. (see [8], [2], [3], [4]).

Definition 1.3

The (Gaussian) hypergeometric function $F(a,b;c;z)$ is defined as,

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (1.3)$$

where, $c \neq 0, -1, -2, \dots$ and $(x)_n$ is the Pochhammer symbol defined using Gamma functions,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)(x+2) \dots (x+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, 3, \dots\} \end{cases}$$

Note:

$(a)_n = (a+n-1)(a)_{n-1}$, $(a)_n = a(a+1)_{n-1}$ and $(1)_n = n!$ for all $n \geq 1$.

We note that, $F(a, b; c; 1)$ converges for $\operatorname{Re}(c-a-b) > 0$ (see [1] [9]) and related to Gamma function defined by,

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (1.4)$$

Ruscheweyh and Singh [10] and others used continued fractions to find sufficient conditions for $zF(a, b; c; z)$ to be in $S^*(\alpha)$ for various choices of the parameters a, b and c . Silverman [11] gave necessary and sufficient condition for $zF(a, b; c; z)$ to be in $S^*(\alpha)$, the class of functions starlike of order α and $K(\alpha)$, the class of functions convex of order α .

We shall introduce a new class of univalent functions:

Definition 1.4

The function $f(z)$ defined by Equation (1.1), is said to be in class $S(\alpha, \beta, \lambda)$, $(-1 \leq \lambda \leq 1, \alpha, \beta \geq 0$ and $z \in U)$, if and only if,

$$\operatorname{Re} \left\{ \frac{zf'(z) + \alpha z^2 f''(z) + \beta z^3 f'''(z)}{f(z)} - \lambda \right\} \geq \left| \frac{zf'(z) + \alpha z^2 f''(z) + \beta z^3 f'''(z)}{f(z)} - 1 \right| \quad (1.5)$$

When $\beta = 0$ the class $S(\alpha, 0, \lambda) \equiv S(\lambda, \alpha)$ as defined by B. Srutha Keerthi, B. Adolf Stephen, A. Gangadharan and S. Sivasubramanian [1].

- For $\alpha = \beta = 0$, the class $S(0, 0, \lambda)$ is class of uniformly starlike function of order λ , $S_p(\lambda)$. (see [9]).
- We let, $S_T(\alpha, \beta, \lambda) = S(\alpha, \beta, \lambda) \cap T$.

II. Main Result

In the present paper, we determine sufficient condition for $zF(a, b; c; z)$ to be in the class $S(\alpha, \beta, \lambda)$ and also give necessary and sufficient condition for $zF(a, b; c; z)$ to be in the class $S_T(\alpha, \beta, \lambda)$ with appropriate restrictions on a, b, c . Furthermore we consider an integral operator related to the hypergeometric function.

To establish main result we need following Lemmas.

LEMMA 2.1

The function $f(z)$ defined by Equation (1.1) is in the class, $S(\alpha, \beta, \lambda)$, $(-1 \leq \lambda \leq 1, \alpha, \beta \geq 0$ and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 3 + \lambda) |a_n| \leq 1 - \lambda \quad (2.1)$$

PROOF:

By using Equation (1.1), we can write,

$$\frac{zf'(z) + \alpha z^2 f''(z) + \beta z^3 f'''(z)}{f(z)} = \frac{Z + \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2)) a_n z^n}{z + \sum a_n z^n} \quad (2.2)$$

Substituting right hand side of above Equation (2.2) in Inequality (1.5),

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Z + \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2)) a_n z^n}{z + \sum a_n z^n} - \lambda \right\} \\ \geq \left| \frac{Z + \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2)) a_n z^n}{z + \sum a_n z^n} - 1 \right| \end{aligned}$$

that is,

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Z + \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2)) a_n z^n}{z + \sum a_n z^n} - 1 \right\} + 1 - \lambda \\ \geq \left| \frac{Z + \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2)) a_n z^n}{z + \sum a_n z^n} - 1 \right| \end{aligned}$$

Using the fact that, $\operatorname{Re}(z) \leq |z|$ and as $f(z)$ is defined in unit disc $U = \{z : |z| < 1\}$, letting $z \rightarrow 1^-$ we get,

$$\frac{2 \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2) - 1) |a_n|}{1 + \sum |a_n|} \leq 1 - \lambda \quad (2.3)$$

Upon simplifying we get,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 3 + \lambda) |a_n| \leq 1 - \lambda$$

which is the required result.

For $\beta = 0$ we get results of B. Srutha Keerthi, B. Adolf Stephen, A. Gangadharan and S. Sivasubramanian [1], [Lemma 2.1, p497].

COROLLARY 2.1

The function $f(z)$ defined by Equation (1.1) is in the class, $S(\alpha, 0, \lambda)$, $(-1 \leq \lambda \leq 1, \alpha \geq 0$ and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) - 3 + \lambda) |a_n| \leq 1 - \lambda$$

LEMMA 2.2

The function $g(z)$ defined by Equation(1.2) is in the class $S_T(\alpha, \beta, \lambda)$ $(-1 \leq \lambda \leq 1, \alpha, \beta \geq 0$ and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 1 - \lambda) b_n \leq 1 - \lambda \quad (2.4)$$

Prof of this Lemma is easily obtained by appropriate rearrangements in Inequality (2.1) above. As stated above, for $\beta = 0$ we get results of [1], [see Lemma 2.2, p497].

COROLLARY 2.2

The function $g(z)$ defined by Equation (1.2) is in the class $S_T(\alpha, 0, \lambda)$ $(-1 \leq \lambda \leq 1, \alpha$, and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) - 1 - \lambda) b_n \leq 1 - \lambda$$

Using Lemma 2.1 we derive the following theorem for (Gaussian) hypergeometric function.

THEOREM 2.1

If $a, b > 0, c > a+b+3$, then the function, $zF(a, b; c; z)$ is in the class $S(\alpha, \beta, \lambda)$ if and only if,

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{(1-\lambda)} \left(\frac{ab(1+2\alpha)}{(c-a-b-1)} + \frac{(\alpha+3\beta)(a)_2(b)_2}{(c-a-b-2)_2} + \frac{\beta(a)_3(b)_3}{(c-a-b-2)_3} \right) + 1 \right] \leq 2$$

PROOF:

From the Equation (1.3)

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n$$

$$zF(a, b; c; z) = z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^{n+1} = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n$$

Therefore $zF(a, b; c; z)$ is in the class $S(\alpha, \beta, \lambda)$ if and only if, (using Equation (2.1)),

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 3 + \lambda) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq 1 - \lambda \quad (2.5)$$

L.H.S. of above Equation (2.5),

$$= \sum_{n=1}^{\infty} (2(n+1) + 2\alpha n(n+1) + 2\beta(n+1)n(n-1) - 3 + \lambda) \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$\leq 2(1+2\alpha) \sum_{n=1}^{\infty} n \frac{(a)_n(b)_n}{(c)_n(1)_n} + 2(\alpha+3\beta) \sum_{n=1}^{\infty} n(n-1) \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$+ 2\beta \sum_{n=1}^{\infty} n(n-1)(n-2) \frac{(a)_n(b)_n}{(c)_n(1)_n} + (1-\lambda) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

Using Definition (1.3) and simplifying we get,

$$\leq \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\left(\frac{2ab(1+2\alpha)}{(c-a-b-1)} + \frac{2(\alpha+3\beta)ab(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} \right. \right.$$

$$\left. \left. + \frac{2\beta ab(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} \right) + 1 - \lambda \right] - (1-\lambda)$$

Which bounded above by $1-\lambda$, which is true if and only if,

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{(1-\lambda)} \left(\frac{ab(1+2\alpha)}{(c-a-b-1)} + \frac{(\alpha+3\beta)ab(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} \right. \right.$$

$$\left. \left. + \frac{\beta ab(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} \right) + 1 \right] \leq 2$$

that is,

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{(1-\lambda)} \left(\frac{ab(1+2\alpha)}{(c-a-b-1)} + \frac{(\alpha+3\beta)(a)_2(b)_2}{(c-a-b-2)_2} + \frac{\beta(a)_3(b)_3}{(c-a-b-2)_3} \right) + 1 \right] \leq 2$$

which proves the theorem.

Clearly for $\beta = 0$ we get Theorem 2.1 of [1], p499.

COROLLARY 2.3

If $a, b > 0, c > a+b+3$, then the function, $zF(a, b; c; z)$ is in the class $S(\alpha, 0, \lambda)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 + \frac{2ab(1+2\alpha)(c-a-b-2) + \alpha(a+1)(b+1)}{(1-\lambda)(c-a-b-1)(c-a-b-2)} \right] \leq 2$$

Similarly using Lemma 2.2, we derive the following theorem for (Gaussian) hypergeometric function.

THEOREM 2.2

If $a, b > -1, ab < 0, c > a+b+3$, then the function, $zF(a, b; c; z)$ is in the class $S_T(\alpha, \beta, \lambda)$ if and only if,

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2(1+2\alpha)}{(c-a-b-1)} + \frac{2(\alpha+3\beta)(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} + \frac{2\beta(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} + \frac{1-\lambda}{ab} \right] \leq 0$$

that is,

$$\begin{aligned} & -2 \left[\frac{\beta(a)_3(b)_3 + (\alpha+3\beta)(a)_2(b)_2(c-a-b-3) + ab(1+2\alpha)(c-a-b-3)_2}{1-\lambda} \right] \\ & \leq (c-a-b-3)_3 \end{aligned} \quad (2.6)$$

PROOF:

Using the conditions on a, b, c and Equation (1.3), we have,

$$zF(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n$$

Therefore $zF(a, b; c; z)$ is in the class $S_T(\alpha, \beta, \lambda)$ if, (using Equation (2.4))

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 1 - \lambda) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left| \frac{c}{ab} \right| (1-\lambda)$$

that is,

$$\sum_{n=0}^{\infty} (2(n+2) + 2\alpha(n+2)(n+1) + 2\beta(n+1)(n+2) - 1 - \lambda) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \leq \left| \frac{c}{ab} \right| (1-\lambda)$$

that is,

$$\begin{aligned} & \sum_{n=0}^{\infty} [(2+4\alpha)(n+1) + (2\alpha+6\beta)n(n+1) + 2\beta(n+1)n(n-1) + 1 - \lambda] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ & \leq \left| \frac{c}{ab} \right| (1-\lambda) \end{aligned}$$

By using Definition 1.3 (Equation 1.4) and simplifying we get,

$$\begin{aligned} & \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2(1+2\alpha)}{(c-a-b-1)} + \frac{2(\alpha+3\beta)(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} + \frac{2\beta(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} + \frac{1-\lambda}{ab} \right] \\ & - \frac{c}{ab} (1-\lambda) \leq \left| \frac{c}{ab} \right| (1-\lambda) \end{aligned}$$

that is,

$$\begin{aligned} & \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2(1+2\alpha)}{(c-a-b-1)} + \frac{2(\alpha+3\beta)(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} + \frac{2\beta(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} + \frac{1-\lambda}{ab} \right] \\ & + \left| \frac{c}{ab} \right| (1-\lambda) \leq \left| \frac{c}{ab} \right| (1-\lambda) \end{aligned}$$

that is,

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2(1+2\alpha)}{(c-a-b-1)} + \frac{2(\alpha+3\beta)(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} + \frac{2\beta(a+1)(b+1)(a+2)(b+2)}{(c-a-b-1)(c-a-b-2)(c-a-b-3)} + \frac{1-\lambda}{ab} \right] \\ & \leq 0 \end{aligned}$$

that is,

$$-2 \left[\frac{\beta(a)_3(b)_3 + (\alpha + 3\beta)(a)_2(b)_2(c - a - b - 3) + (1 + 2\alpha)ab(c - a - b - 3)_2}{1 - \lambda} \right] \leq (c - a - b - 3)_3.$$

Which proves the theorem.

Clearly for $\beta = 0$ we get Theorem 2.2 of [1], p500.

III. Integral Operator

In this section, we obtain similar type of results in connection with a particular integral operator $G(a, b; c; z)$ acting on $F(a, b; c; z)$ as follows:

$$G(a, b; c; z) = \int_0^z F(a, b; c; t) dt = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n \quad (3.1)$$

THEOREM 3.1

If $a, b > 1$, and, $c > a + b + 3$, then the function, $G(a, b; c; z)$ defined by Equation 3.1 is in the class $S(\alpha, \beta, \lambda)$ if and only if,

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[2 + \frac{2\alpha ab}{(c-a-b-1)} + \frac{2\beta ab(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} - \frac{(3-\lambda)(c-a-b)}{(a-1)(b-1)} \right] + \frac{(3-\lambda)(c-1)}{(a-1)(b-1)} \leq 2(1-\lambda) \quad (3.2)$$

PROOF:

By Lemma 2.1, $G(a, b; c; z)$ defined by Equation 3.1 is in the class $S(\alpha, \beta, \lambda)$ if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 3 + \lambda) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \right| \leq 1 - \lambda \quad (3.3)$$

Left Hand side of above Equation 3.3

$$\begin{aligned} &= \sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 3 + \lambda) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \right| \\ &= 2 \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} + 2\alpha \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} + 2\beta \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+2}} - (3-\lambda) \left[\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} - 1 \right] \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[2 + \frac{2\alpha ab}{(c-a-b-1)} + \frac{2\beta ab(a+1)(b+1)}{(c-a-b-1)(c-a-b-2)} - \frac{(3-\lambda)(c-a-b)}{(a-1)(b-1)} \right] \\ &\quad + \frac{(3-\lambda)(c-1)}{(a-1)(b-1)} - (1-\lambda) \end{aligned}$$

Which is bounded above by $(1-\lambda)$, hence $G(a, b; c; z)$ is in the class $S(\alpha, \beta, \lambda)$ if and only if, Inequality 3.2 is satisfied, that proves the theorem.

THEOREM 3.2

If $a, b > -1, ab < 0, c > a+b+3$, then the function, $G(a, b; c; z)$ defined by Equation 3.1 is in the class $S_T(\alpha, \beta, \lambda)$ if and only if,

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{ab} (c-a-b-1) + 2\alpha + \frac{2\beta(a+1)(b+1)}{(c-a-b-2)} - \frac{(1+\lambda)(c-a-b-1)_2}{(a-1)_2(b-1)_2} \right] + \frac{(1+\lambda)(c-1)_2}{(a-1)_2(b-1)_2} \leq 0 \quad (3.4)$$

PROOF:

Using the conditions on a, b, c we have,

$$G(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} z^n = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} z^n$$

Therefore $G(a, b; c; z)$ is in the class $S_T(\alpha, \beta, \lambda)$ if and only if, (using Lemma 2.2)

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) + 2\beta n(n-1)(n-2) - 1 - \lambda) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \leq \left| \frac{c}{ab} \right| (1-\lambda)$$

Left hand side of above inequality is

$$= \sum_{n=0}^{\infty} (2(n+2) + 2\alpha(n+2)(n+1) + 2\beta n(n+1)(n+2) - 1 - \lambda) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}$$

$$\begin{aligned}
 &= 2 \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + 2\alpha \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + 2\beta \sum_{n=1}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-1}} \\
 &\quad - (1+\lambda) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\
 &= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{ab} (c-a-b-1) + 2\alpha + \frac{2\beta(a+1)(b+1)}{(c-a-b-2)} \right. \\
 &\quad \left. - \frac{(1+\lambda)(c-a-b)(c-a-b-1)}{ab(a-1)(b-1)} \right] + \frac{(1+\lambda)c(c-1)}{ab(a-1)(b-1)} - (1-\lambda) \frac{c}{ab} \\
 &= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{2}{ab} (c-a-b-1) + 2\alpha + \frac{2\beta(a+1)(b+1)}{(c-a-b-2)} \right. \\
 &\quad \left. - \frac{(1+\lambda)(c-a-b)(c-a-b-1)}{ab(a-1)(b-1)} \right] + \frac{(1+\lambda)c(c-1)}{ab(a-1)(b-1)} + (1-\lambda) \left| \frac{c}{ab} \right|
 \end{aligned}$$

Which is bounded above by $\left| \frac{c}{ab} \right| (1-\lambda)$, hence we get the inequality 3.4, which completes the proof.

Remarks:

In above results,

- For $\beta = 0$, we get results of B. Srutha Keerthi, B. Adolf Stephen, A. Gangadharan and S. Sivasubramanian of [1].
- For choice of $\alpha = \beta = 0$, we get results of N.E. Cho, S.Y. Woo and S. Owa [9].

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