



Conical representation of Rational Quartic Trigonometric Bèzier curve with two shape parameters

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Abstract: A rational quartic trigonometric Bèzier curve with two shape parameters, which is analogous to quadratic Bèzier curve, is presented in this paper. The shape of the curve can be adjusted as desired, by simply altering the value of shape parameters, without changing the control polygon. The rational quartic trigonometric Bèzier curve can be made close to the rational quadratic Bèzier curve or closer to the given control polygon than the rational quadratic Bèzier curve. The ellipses and circles can be representation exactly by using rational quartic trigonometric Bèzier curve.

Keywords: Rational Trigonometric Bèzier Basis Function, Rational Trigonometric Bèzier Curve, Shape Parameter, Open curves, Close curves.

I. Introduction

Rational spline is a commonly used spline function. In many cases the rational spline curves better approximating functions than the usual spline functions. It has been observed that many simple shapes including conic section and quadric surfaces can not be represented exactly by piecewise polynomials, whereas rational polynomials can exactly represent all conic sections and quadric surfaces in an easy manner (see Sarfraz, M. And Habib, Z. [11]). Many authors have studied this spline, especially the rational cubic spline see; Duan, Q., Wang, L., Twizell, E.H. [3], Duan, Q., Zhang, H., Zhang, Y., Twizell, E.H. [4]

Recently, many papers investigate the trigonometric Bèzier-like polynomial, trigonometric spline and their applications. Many kinds of methods based on trigonometric polynomials were also established for free form curves and surfaces modeling (see, Han, Xi-An, Ma-Chen, Y. and Huang, X. [8], Dube, M. and Yadav, B., [5], Han, X., ([6], [7]), Abbas M, Yahaya SH, Majid AA, Ali JM [1], I. J. Schoenberg [9], Yang L, Li J, Chen G, ([13], [14])). Liu, H., Li, L., and Zhang, D. [10] presented a study on class of TC-Bèzier curve with shape parameters. Bashir, U., Abbas, M., Ahmad, Abd Majid, Ali, J. M., [2] have discussed the rational quadratic trigonometric Bèzier curve with two shape parameters. Base on this idea, we have constructed the Rational Quartic Trigonometric Bèzier curve with two shape parameters.

In this paper, rational quartic trigonometric Bèzier curve with two shape parameters is presented. The purposed curve enjoys all the geometric properties of the traditional rational quartic Bèzier curve. The local control on the shape of the curve can be attained by altering the values of the shape parameters as well as the weight. The curve exactly represents some quartic trigonometric curves such as the arc of an ellipse and the arc of a circle and best approximates the ordinary rational quadratic Bèzier curve. The paper is organized as follows. In section 2, the asis functions of the quartic trigonometric Bèzier curve with two shape parameters are established and the properties of the basis function has been described. In section 3, rational quartic trigonometric Bèzier curves and their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and explained by using figures. In section 5, the representation of ellipse and circle are given. In section 6, the approximation of the rational quartic trigonometric Bèzier curve to the ordinary rational quadratic Bèzier curve is presented.

II. Quartic Trigonometric Bèzier Basis Functions

In this section, definition and some properties of quartic trigonometric Bèzier basis functions with two shape parameters are given as follows:

Definition 2.1: For two arbitrarily real value of λ and μ where $\lambda, \mu \in [0,1]$ the following three functions of $t(t \in [0,1])$ are defined as quartic trigonometric Bèzier basis functions with two shape parameter λ, μ :

$$\begin{cases} b_0(t) = \left(1 - \sin \frac{\pi}{2} t\right) \left(1 - \lambda \sin \frac{\pi}{2} t\right)^3 \\ b_1(t) = 1 - b_0 - b_2 \\ b_2(t) = \left(1 - \cos \frac{\pi}{2} t\right) \left(1 - \mu \cos \frac{\pi}{2} t\right)^3 \end{cases} \quad (2.1)$$

For $\lambda = \mu = 0$, the basis functions are linear trigonometric polynomials. For $\lambda, \mu \neq 0$, the basis functions are quartic trigonometric polynomials.

Theorem 2.1: The basis functions (2.1) have the following properties:

- (a) **Nonnegativity:** $b_i(t) \geq 0, i = 0, 1, 2$.
- (b) **Partition of unity:** $\sum_{i=0}^2 b_i(t) = 1$
- (c) **Monotonicity:** For a given parameter t , $b_0(t)$ is monotonically decreasing λ and μ respectively; $b_2(t)$ is monotonically increasing for the shape parameters λ and μ respectively.

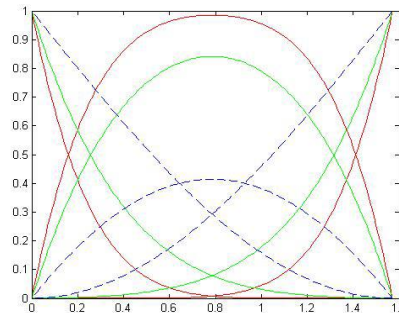


Figure 1: The quartic trigonometric basis function.

- (d) **Symmetry:** For $b_i(t; \lambda, \mu) = b_{2-i}(1-t; \mu, \lambda)$, for $i = 0, 1, 2$.

Proof: (a) For $t \in [0, 1]$ and $\lambda, \mu \in [0, 1]$, then

$$\left(1 - \sin \frac{\pi}{2} t\right) \geq 0, \left(1 - \lambda \sin \frac{\pi}{2} t\right) \geq 0,$$

$$\left(1 - \lambda \sin \frac{\pi}{2} t\right)^3 \geq 0, \sin \frac{\pi}{2} t \geq 0,$$

$$\left(1 - \cos \frac{\pi}{2} t\right) \geq 0, \left(1 - \mu \sin \frac{\pi}{2} t\right) \geq 0,$$

$$\left(1 - \mu \cos \frac{\pi}{2} t\right)^3 \geq 0, \cos \frac{\pi}{2} t \geq 0, \lambda \geq 0, \mu \geq 0.$$

It is obvious that $b_i(t) \geq 0, i = 0, 1, 2$.

$$\begin{aligned} \text{(b)} \quad \sum_{i=0}^2 b_i(t) &= \left(1 - \sin \frac{\pi}{2} t\right) \left(1 - \lambda \sin \frac{\pi}{2} t\right)^3 + \left[1 - \left(1 - \sin \frac{\pi}{2} t\right) \left(1 - \mu \sin \frac{\pi}{2} t\right)^3 - \left(1 - \cos \frac{\pi}{2} t\right) \left(1 - \mu \cos \frac{\pi}{2} t\right)^3\right] \\ &\quad + \left(1 - \cos \frac{\pi}{2} t\right) \left(1 - \mu \cos \frac{\pi}{2} t\right)^3 = 1. \end{aligned}$$

Fig.1. shows the curves of the rational quartic trigonometric basis functions for $\lambda = \mu = 1$ (red solid), $\lambda = \mu = 0.5$ (green solid), $\lambda = \mu = 0$ (blue dashed).

III. Rational Quartic trigonometric Bèzier curve

We construct the rational quartic trigonometric Bèzier curve with two shape parameters as follows:

Definition 3.1: The Rational Quartic Trigonometric Bèzier curve with two shape parameters is defined as:

$$C(t) = \frac{b_0 P_0 + v b_1 P_1 + b_2 P_2}{b_0 + v b_1 + b_2} \quad (3.1)$$

$t \in [0, 1], \lambda, \mu \in [0, 1]$, where $P_i (i = 0, 1, 2)$ in R^2 or R^3 the control points, $b_i (i = 0, 1, 2)$ are the basis functions defined in (2.1) and v is scalar, called that weight of function. We assume that $v \geq 0$. If $v = 1$, we get a non-rational trigonometric Bèzier curve, since the denominator is identically equal to one.

The curve defined by (3.1) possesses some properties which can be obtained easily from the properties of the basis function.

Theorem 3.1: The Rational Quartic trigonometric Bèzier curve (3.1) have the following properties:

- (a) **End point properties:**

$$C(0) = P_0, \quad C(1) = P_2,$$

$$C'(0) = \frac{\pi}{2} (1 + 3\lambda) (P_1 - P_0) v,$$

$$C'(1) = \frac{\pi}{2} (1 + 3\mu) (P_2 - P_1) v$$

$$C''(0) = \frac{\pi^2}{4} [v(2\lambda(2 + 3\lambda) + (1 - \mu)^2(1 + \mu) + 2(1 + 3\lambda)^2(v - 1))(P_1 - P_0) + (1 - \mu)^2(1 + \mu)(P_2 - P_0)],$$

$$C''(1) = \frac{\pi^2}{4} [v(2\mu(2 + 3\mu) + (1 - \lambda)^2(1 + \lambda) + 2(1 + 3\mu)^2(v - 1))(P_2 - P_1) + (1 - \lambda)^2(1 + \lambda)(P_0 - P_2)]$$

- (b) **Symmetry:** P_0, P_1, P_2 and P_2, P_1, P_0 define the same curve in different parametrizations, that is

$$C(t; \lambda, \mu; P_0, P_1, P_2) = C(1 - t; \mu, \lambda; P_2, P_1, P_0),$$

$$t \in [0, 1], \lambda, \mu \in [0, 1].$$

(c) **Geometric invariance:** The shape of the curve (3.1) is independent of the choice of coordinates, i.e., it satisfies the following two equations:

$$C(t; \lambda, \mu; P_0 + q, P_1 + q, P_2 + q) = C(t; \mu, \lambda; P_0, P_1, P_2) + q$$

$$C(t; \lambda, \mu; P_0 * T, P_1 * T, P_2 * T) = C(t; \mu, \lambda; P_0, P_1, P_2) * T$$

where q is an arbitrary vector in R^2 and R^3 and T is an arbitrary $d \times d$ matrix, $d = 2$ or 3 .

(d) **Convex hull property:** From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

IV. Shape control of the rational quartic Trigonometric Bèzier curve

The parameters λ and μ controls the shape of the curve (3.1). In figure 2, The rational quartic trigonometric Bèzier curve $C(t)$ gets closer to the control polygon as the values of the parameters λ and μ increases. In figure 2, the curves are generated by setting the values of λ, μ as $\lambda = \mu = 0, v = 2$ (red solid lines) $\lambda = \mu = 0.5, v = 3$ (blue dashed lines), $\lambda = \mu = 1, v = 4$ (green solid lines).

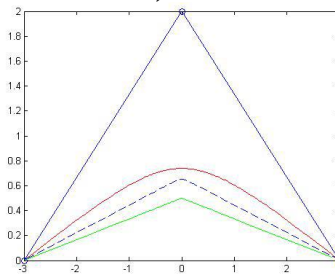


Figure 2: The effect on the shape of rational quartic trigonometric Bèzier curves with altering the values of λ and μ simultaneously.

V. The representation of Ellipse

Theorem 5.1: Let P_0, P_1 and P_2 be three control points on an ellipse with semi axes $\sqrt{2a}$ and $\frac{1}{\sqrt{2}}b$, by the proper selection of coordinates, their coordinates can be written in the form

$$P_0 = \begin{pmatrix} -a \\ b \end{pmatrix}, P_1 = \begin{pmatrix} 0 \\ b \end{pmatrix}, P_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

Then the corresponding rational quartic trigonometric Bèzier curve with the shape parameters $\lambda = \mu = 0$ with $v = \frac{1}{2}$ and local domain $t \in [t_1, t_2]$ represents arc of an ellipse with

$$\begin{cases} x(t) = a \left(\cos \frac{\pi}{2}(t) - \sin \frac{\pi}{2}(t) \right) \\ y(t) = \frac{1}{2} b \left(\sin \frac{\pi}{2}(t) + \cos \frac{\pi}{2}(t) - 1 \right) \end{cases} \quad (5.1)$$

Where $0 \leq t_1, t_2 \leq 4$.

Proof: If we take $\lambda = \mu = 0, v = \frac{1}{2}$ and

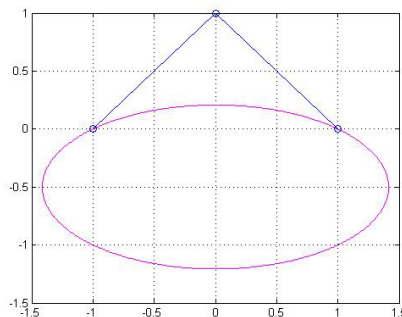


Figure 3: The representation of Ellipse

$$P_0 = \begin{pmatrix} -a \\ b \end{pmatrix}, P_1 = \begin{pmatrix} 0 \\ b \end{pmatrix}, P_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

into (3.1), then the coordinates of rational quartic trigonometric Bèzier curve are

$$\begin{cases} x(t) = a \left(\sin \frac{\pi}{2}(t) - \cos \frac{\pi}{2}(t) \right) \\ y(t) = \frac{1}{2} b \left(\sin \frac{\pi}{2}(t) + \cos \frac{\pi}{2}(t) - 1 \right) \end{cases}$$

This gives the intrinsic equation

$$\left(\frac{x(t)}{\sqrt{2a}}\right)^2 + \left(\frac{y(t) + \frac{1}{2}b}{\frac{1}{\sqrt{2}}b}\right)^2 = 1.$$

It is an equation of an ellipse.

Fig.3 shows the Ellipse.

Corollary 5.2: According to theorem (5.1), if, $a = \frac{1}{\sqrt{2}}, b = \sqrt{2}$ then the corresponding Rational Quartic trigonometric Bèzier curve with the shape parameter $\lambda = \mu = 0$ and local domain $t \in [0, 4]$ represents arc of the circle.

Fig.4 shows the Circle.

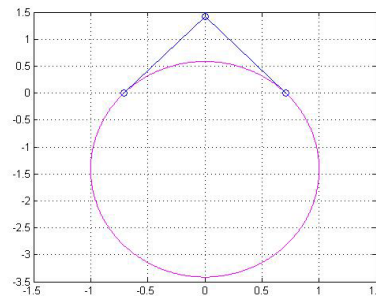


Figure 4: The representation of Circle

VI. Approximability

Control polygons play an important role in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relation of the rational quartic trigonometric Bèzier curves and rational quadratic Bèzier curves with same control points.

Theorem 6.1: Suppose P_0, P_1 and P_2 are not collinear; the relationship between rational quartic trigonometric Bèzier curves $C(t)$ (3.12) and rational quadratic Bèzier curve.

$$B(t) = \frac{\sum_{i=0}^2 P_i B_i(t) w_i}{\sum_{i=0}^2 B_i(t) w_i} \quad (6.1)$$

where $B_i(t) = \sum_{i=0}^2 \binom{2}{i} (1-t)^{2-i} t^i$, ($i = 0, 1, 2$) are the Bernstein polynomials and the scalar w are the weights, with the same control points P_i ($i = 0, 1, 2$) are as follows:

$$C(0) = B(0), \quad C(1) = B(1) \\ C\left(\frac{1}{2}\right) - P_1 = \frac{2(1+\nu)}{2(1-\nu) + k\nu} \left(B\left(\frac{1}{2}\right) - P_1\right) \quad (6.2)$$

where $k = \left[(1 - \sin \frac{\pi}{2} t)(1 - \lambda \sin \frac{\pi}{2} t)^3\right]^{-1}$ with the assumption that $\lambda = \mu$.

Proof: We assume that $w_0 = w_2 = 1$ and $w_1 = \nu$ then the ordinary rational quadratic Bèzier curve (6.1) takes the form

$$B(t) = \frac{(1-t)^2 P_0 + 2(1-t)t P_1 \nu + t^2 P_2}{(1-t)^2 + 2(1-t)t\nu + t^2}$$

By simple computation, then

$$C(0) = P_0 = B_0, \quad C(1) = P_2 = B(1) \text{ and} \\ B\left(\frac{1}{2}\right) - P_1 = \frac{1}{2(1+\nu)} (P_0 - 2P_1 + P_2) \quad (6.3)$$

For $\lambda = \mu$,

$$C\left(\frac{1}{2}\right) - P_1 = \frac{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{\lambda}{\sqrt{2}}\right)^3 (P_0 - 2P_1 + P_2)}{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{\lambda}{\sqrt{2}}\right)^3 (2 - 2\nu) + \nu}$$

Let $= \left[(1 - \sin \frac{\pi}{2} t)(1 - \lambda \sin \frac{\pi}{2} t)^3\right]^{-1}$, then

$$C\left(\frac{1}{2}\right) - P_1 = \frac{(P_0 - 2P_1 + P_2)}{2(1-\nu) + k\nu}$$

$$C\left(\frac{1}{2}\right) - P_1 = \frac{2(1+\nu)}{2(1-\nu) + k\nu} \left(B\left(\frac{1}{2}\right) - P_1 \right).$$

Then (6.2) holds.

Corollary 6.1: The rational quartic trigonometric Bèzier curve is closer the control polygon than the rational quadratic Bèzier curve if and only if $\frac{\sqrt{2}(\sqrt{2}-1)^{\frac{1}{3}}-1}{(\sqrt{2}-1)^{\frac{1}{3}}} \leq \lambda, \mu \leq 1$.

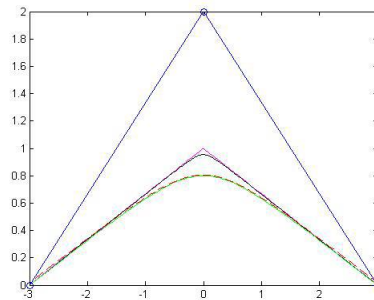


Figure 5: the relationship between the rational quartic trigonometric Bèzier curve and rational quadratic Bèzier curve.

Corollary 6.2: When $\lambda = \mu = \frac{\sqrt{2}(\sqrt{2}-1)^{\frac{1}{3}}-1}{(\sqrt{2}-1)^{\frac{1}{3}}}$, the rational quartic trigonometric Bèzier curve is closer to the rational quadratic Bèzier curve, i.e. $C\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right)$.

Fig.5 shows the relationship between the rational quartic trigonometric Bèzier curve and rational quadratic Bèzier curve. The quartic trigonometric Bèzier curve is closer to rational quadratic Bèzier curve. The rational quartic trigonometric Bèzier curve (blue dashed) with parameter $\lambda = \mu = \frac{\sqrt{2}(\sqrt{2}-1)^{\frac{1}{3}}-1}{(\sqrt{2}-1)^{\frac{1}{3}}}$ is analogous to ordinary quadratic Bèzier curve (green solid).

VII. Conclusion

In this paper, we have presented the rational quartic trigonometric Bèzier curve with two shape parameters. Each section of the curve only refers to the three control points. We can design different shape curves by changing parameters. The curve represent ellipse and circle when adjusting the control points and parameter value.

References

- [1] Abbas M, Yahaya SH, Majid AA, Ali JM, Spur gear tooth design with shaped transition curve using T-B'ezier function. *Procedia Eng* 50, 2012, 211-21.
- [2] Bashir, U., Abbas, M., Ahmad, Abd Majid, Ali, J. M., The rational quadratic trigonometric B'ezier curve with two shape parameters, Ninth International Conference on computer graphics, Imaging and Visualization, IEEE, (2012), 31-36.
- [3] Duan, Q., Wang, L., Twizell, E.H., A new C2 rational interpolation based on function values and constrained control of the interpolant curves, *Applied Mathematics and Computation* 161, (2005), 311-322.
- [4] Duan, Q., Wang, L., Twizell, E.H., A new weighted rational cubic interpolation and its approximation, *Applied Mathematics and Computation*, 168, (2005), 990-1003.
- [5] Dube, M., Yadav, B., On shape control of a Quartic trigonometric B'ezier curve with a shape parameter. *International Journal of Advanced Research*, 2013, vol.2, Issue 1, 648-654.
- [6] Han, X., Piecewise quadratic trigonometric polynomial curves. *Mathematics of Computation*. 2003, 72(243):1369-1378.
- [7] Han, X., Cubic trigonometric polynomial curves with a shape parameter. *Computer Aided Geometric Design*, 2004, 21(6):535-548.
- [8] Han, X.A., Y.C. Ma, and X.L. Huang, The cubic trigonometric B'ezier curves with two shape parameters. *Applied Mathematics Letters*, 2009, 22(2):226-231.
- [9] I. J. Schoenberg, On trigonometric spline interpolation. *Journal Math Mech*, 1964, 13:795-825.
- [10] Liu, H., L. Li, and D. Zhang. Study on a Class of TC-B'ezier curves with shape parameters. *Journal of Information and Computational Science*, 2011, 8(7): 1217-1223.
- [11] Sarfraz, M. And Habib, Z., Rational cubic and conics representation: A practical approach, *IIUM, Engineering Journal*, 1(2), (2000), 1-9.
- [12] Wei X. Xu, L. Qiang Wang, Xu Min Liu, Quadratic TC-B'ezier curves with shape parameter. *Advanced Materials Research*, 2011, (179- 180):1187-1192.
- [13] Yang L, Li J, Chen G, A class of quasi-quartic trigonometric B'ezier curve and surfaces. *Journal of Information and Computational Science*, 2012, 7, 72-80.
- [14] Yang L, Li J, Chen G, Trigonometric extension of quartic B'ezier curves. *IEEE International Conference on Multimedia Technology (ICMT)*, 2011, 926-929.