



SOME FIXED POINT THEOREMS IN HILBERT SPACE

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Abstract: In the present paper, we find some fixed point theorems in Hilbert space satisfying rational type contractive condition. Our result is extension and generalization of many previous known results.

Keywords: Fixed point, Closed subset, Hilbert space, Cauchy sequence.

I. Introduction

After the Banach's fixed point theorems many researchers worked on Hilbert spaces for generalizing this principle. Some generalization of Banach fixed point theorems were given by D.S. Jaggi [1], Fisher [2], Khare [3]. Ganguly and Bandyopadhyay [4], Koparde and waghmode [5], Pandhare [6], Veerapandi and Anil Kumar [7] investigated the properties of fixed points of family of mappings on complete metric spaces and in Hilbert spaces.

Kannan [8] proved that a self-mapping T on complete metric space (X, d) satisfying the condition $d(x, y) \leq \alpha\{d(x, Tx) + d(y, Ty)\}$

For all $x, y \in X$ where $0 < \alpha < \frac{1}{2}$ has a unique fixed in (X, d) .

Koparde and Wghmode [9] have proved fixed point theorem for a self-mapping T on a closed subset S of Hilbert space H , satisfying the Kannan type condition

$$\|Tx - Ty\|^2 \leq \alpha\{\|x - Tx\|^2 + \|y - Ty\|^2\}$$

For all $x, y \in S$ and $x \neq y$; $0 \leq \alpha < \frac{1}{2}$

II. Main Results

Theorem 2.1: Let C be a closed subset of Hilbert space X and T be a mapping on X into it-self satisfying

$$\|Tx - Ty\|^2 \leq \alpha \frac{\|y - Ty\|^2 + \|x - Tx\|^2 + \|y - Tx\|^2}{1 + \|x - Tx\|^2 \|x - Ty\|^2} + \beta\{\|y - Ty\|^2 + \|y - Tx\|^2\} + \gamma\|x - y\|^2 \quad (2.1)$$

For all $x, y \in X$, where α, β and γ are non-negative real with $4\alpha + 3\beta + \gamma < 1$. Then T has a unique fixed point in X .

Proof: For some $x_0 \in X$, we define a sequence $\{x_n\}$ of iterates of T as follows

$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \dots$ i.e $x_{n+1} = Tx_n$, for $n = 0, 1, 2, 3 \dots$ For this consider

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

Then from (2.1)

$$\|x_{n+1} - x_n\|^2 \leq \alpha \frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_n\|^2}{1 + \|x_n - Tx_n\|^2 \|x_n - Tx_{n-1}\|^2} + \beta\{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_{n-1} - Tx_n\|^2\}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha \frac{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_{n+1}\|^2}{1 + \|x_n - x_{n+1}\|^2 \|x_n - x_n\|^2} + \beta\{\|x_{n-1} - x_n\|^2 + \|x_{n-1} - x_{n+1}\|^2\}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\}$$

$$+ \beta\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} + \gamma\|x_n - x_{n-1}\|^2$$

$$\{1 - 2\alpha - \beta\}\|x_{n+1} - x_n\|^2 \leq \{2\alpha + 2\beta + \gamma\}\|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq \frac{2\alpha + 2\beta + \gamma}{1 - 2\alpha - \beta} \|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq k \|x_n - x_{n-1}\|^2$$

Where $k = \frac{2\alpha + 2\beta + \gamma}{1 - 2\alpha - \beta}$. Continuing in the same way, we have

$$\|x_{n+1} - x_n\|^2 \leq k^n \|x_1 - x_0\|^2$$

Where $n = 1, 2, 3, \dots$, For any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\|^2 \leq \|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_{n+2}\|^2 + \dots + \|x_{m-1} - x_m\|^2$$

$$\leq k^n \|x_1 - x_0\|^2 + k^{n+1} \|x_1 - x_0\|^2 + \dots + k^{m-1} \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq k^n (1 + k + \dots + k^{m-n-1}) \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq \frac{k^n}{1 - k} \|x_1 - x_0\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore $\{x_n\}$ is a Cauchy sequence in C . Since C is a closed subset of a Hilbert space X , there exists an element $u \in C$ which is the limit of $\{x_n\}$. i.e.

$$u = \lim_{n \rightarrow \infty} x_n$$

Now, further we have

$$\begin{aligned} \|u - Tu\|^2 &= \|(u - x_n) + (x_n - Tu)\|^2 \\ &\leq \|u - x_n\|^2 + \|x_n - Tu\|^2 + 2\operatorname{Re}\langle u - x_n, x_n - Tu \rangle \\ &\leq \|u - x_n\|^2 + \alpha \frac{\|u - Tu\|^2 + \|x_{n-1} - Tx_{n-1}\|^2 + \|u - Tx_{n-1}\|^2}{1 + \|x_{n-1} - Tx_{n-1}\|^2 \|x_{n-1} - Tu\|^2} + \beta\{\|u - Tu\|^2 + \|u - Tx_{n-1}\|^2\} \\ &\quad + \gamma\|x_{n-1} - u\|^2 + 2\operatorname{Re}\langle u - x_n, x_n - Tu \rangle \end{aligned}$$

Letting $n \rightarrow \infty$, so that $x_{n-1}, x_n \rightarrow u$, we get

$$\|u - Tu\|^2 \leq (3\alpha + 2\beta) \|u - Tu\|^2$$

This implies $u = Tu$, since $3\alpha + 2\beta < 1$. Hence u is a fixed point of self-mapping T .

Uniqueness: Let $v \in C$ be another fixed point of T such that $v \neq u$ then

$$\begin{aligned} \|v - u\|^2 &= \|Tv - Tu\|^2 \\ &\leq \alpha \frac{\|u - Tu\|^2 + \|v - Tv\|^2 + \|u - Tv\|^2}{1 + \|v - Tv\|^2 \|v - Tu\|^2} + \beta\{\|u - Tu\|^2 + \|u - Tv\|^2\} + \gamma\|v - u\|^2 \\ \|v - u\|^2 &\leq (\alpha + \beta + \gamma) \|v - u\|^2 \end{aligned}$$

Which is a contradiction; for $\alpha + \beta + \gamma < 1$. Thus u is a unique fixed point of T in X .

Theorem 2.2: Let C be a closed subset of Hilbert space X and T be a mapping on X into it-self satisfying

$$\|Tx - Ty\|^2 \leq \alpha \frac{\|y - Ty\|^2 + \|x - Tx\|^2 + \|y - Tx\|^2}{1 + \|x - Tx\|^2 \|x - Ty\|^2} + \beta\{\|y - Ty\|^2 + \|y - Tx\|^2\} + \gamma\|x - y\|^2 + \delta \min\{\|x - Ty\|^2, \|y - Tx\|^2\} \quad (2.2)$$

For all $x, y \in X$, where α, β, γ and δ are non-negative real with $4\alpha + 3\beta + \gamma + \delta < 1$. Then T has a unique fixed point in X .

Proof: For some $x_0 \in X$, we define a sequence $\{x_n\}$ of iterates of T as follows

$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \dots$ i.e. $x_{n+1} = Tx_n$, for $n = 0, 1, 2, 3, \dots$. For this consider

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

Then from (2.2)

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &\leq \alpha \frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_n\|^2}{1 + \|x_n - Tx_n\|^2 \|x_n - Tx_{n-1}\|^2} + \beta\{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_{n-1} - Tx_n\|^2\} \\ &\quad + \gamma\|x_n - x_{n-1}\|^2 + \delta \min\{\|x_n - Tx_{n-1}\|^2, \|x_{n-1} - Tx_n\|^2\} \end{aligned}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha \frac{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_{n+1}\|^2}{1 + \|x_n - x_{n+1}\|^2 \|x_n - x_{n-1}\|^2} + \beta \{\|x_{n-1} - x_n\|^2 + \|x_{n-1} - x_{n+1}\|^2\} + \gamma \|x_n - x_{n-1}\|^2 + \delta \min\{\|x_n - x_{n-1}\|^2, \|x_{n-1} - x_{n+1}\|^2\}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} + \beta \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} + \gamma \|x_n - x_{n-1}\|^2$$

$$\{1 - 2\alpha - \beta\} \|x_{n+1} - x_n\|^2 \leq \{2\alpha + 2\beta + \gamma\} \|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq \frac{2\alpha + 2\beta + \gamma}{1 - 2\alpha - \beta} \|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq k \|x_n - x_{n-1}\|^2$$

Where $k = \frac{2\alpha + 2\beta + \gamma}{1 - 2\alpha - \beta}$. Continuing in the same way, we have

$$\|x_{n+1} - x_n\|^2 \leq k^n \|x_1 - x_0\|^2$$

Where $n = 1, 2, 3, \dots$, For any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\|^2 \leq \|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_{n+2}\|^2 + \dots + \|x_{m-1} - x_m\|^2$$

$$\leq k^n \|x_1 - x_0\|^2 + k^{n+1} \|x_1 - x_0\|^2 + \dots + k^{m-1} \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq k^n (1 + k + \dots + k^{m-n-1}) \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq \frac{k^n}{1 - k} \|x_1 - x_0\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore $\{x_n\}$ is a Cauchy sequence in C . Since C is a closed subset of a Hilbert space X , there exists an element $u \in C$ which is the limit of $\{x_n\}$. i.e.

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$$\begin{aligned} \|u - Tu\|^2 &= \|(u - x_n) + (x_n - Tu)\|^2 \\ &\leq \|u - x_n\|^2 + \|x_n - Tu\|^2 + 2\operatorname{Re}\langle u - x_n, x_n - Tu \rangle \end{aligned}$$

$$\leq \|u - x_n\|^2 + \alpha \frac{\|u - Tu\|^2 + \|x_{n-1} - Tx_{n-1}\|^2 + \|u - Tx_{n-1}\|^2}{1 + \|x_{n-1} - Tx_{n-1}\|^2 \|x_{n-1} - Tu\|^2} + \beta \{\|u - Tu\|^2 + \|u - Tx_{n-1}\|^2\} + \gamma \|x_{n-1} - u\|^2 + \delta \min\{\|x_{n-1} - Tu\|^2, \|u - Tx_{n-1}\|^2\} + 2\operatorname{Re}\langle u - x_n, x_n - Tu \rangle$$

Letting $n \rightarrow \infty$, so that $x_{n-1}, x_n \rightarrow u$, we get

$$\|u - Tu\|^2 \leq (3\alpha + 2\beta + \delta) \|u - Tu\|^2$$

This implies $u = Tu$, since $3\alpha + 2\beta + \delta < 1$. Hence u is a fixed point of self-mapping T .

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$$\begin{aligned} \|v - u\|^2 &= \|Tv - Tu\|^2 \\ &\leq \alpha \frac{\|u - Tu\|^2 + \|v - Tv\|^2 + \|u - Tv\|^2}{1 + \|v - Tv\|^2 \|v - Tu\|^2} + \beta \{\|u - Tu\|^2 + \|u - Tv\|^2\} + \gamma \|v - u\|^2 \\ &\quad + \delta \min\{\|v - Tu\|^2, \|u - Tv\|^2\} \end{aligned}$$

$$\|v - u\|^2 \leq (\alpha + \beta + \gamma + \delta) \|v - u\|^2$$

Which is a contradiction; for $\alpha + \beta + \gamma + \delta < 1$. Thus u is a unique fixed point of T in X .

Theorem 2.3: Let C be a closed subset of Hilbert space X and T be a mapping on X into it-self satisfying

$$\|Tx - Ty\|^2 \leq \alpha \frac{\|y - Ty\|^2 + \|x - Tx\|^2 + \|y - Tx\|^2}{1 + \|x - Tx\|^2 \|x - Ty\|^2} + \beta \{\|y - Ty\|^2 + \|y - Tx\|^2\} + \gamma \|x - y\|^2 + \delta \max\{\|x - Ty\|^2, \|y - Tx\|^2\} \quad (2.3)$$

For all $x, y \in X$, where α, β, γ and δ are non-negative real with $4\alpha + 3\beta + \gamma + 2\delta < 1$. Then T has a unique fixed point in X .

Proof: For some $x_0 \in X$, we define a sequence $\{x_n\}$ of iterates of T as follows

$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \dots$ i.e. $x_{n+1} = Tx_n$, for $n = 0, 1, 2, 3, \dots$. For this consider

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

Then from (2.3)

$$\|x_{n+1} - x_n\|^2 \leq \alpha \frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_n\|^2}{1 + \|x_n - Tx_n\|^2 \|x_n - Tx_{n-1}\|^2} + \beta \{\|x_{n-1} - Tx_{n-1}\|^2 + \|x_{n-1} - Tx_n\|^2\} \\ + \gamma \|x_n - x_{n-1}\|^2 + \delta \max\{\|x_n - Tx_{n-1}\|^2, \|x_{n-1} - Tx_n\|^2\}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha \frac{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_{n+1}\|^2}{1 + \|x_n - x_{n+1}\|^2 \|x_n - x_n\|^2} + \beta \{\|x_{n-1} - x_n\|^2 + \|x_{n-1} - x_{n+1}\|^2\} \\ + \gamma \|x_n - x_{n-1}\|^2 + \delta \max\{\|x_n - x_n\|^2, \|x_{n-1} - x_{n+1}\|^2\}$$

$$\|x_{n+1} - x_n\|^2 \leq \alpha \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} \\ + \beta \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} + \gamma \|x_n - x_{n-1}\|^2 + \delta \|x_{n-1} - x_{n+1}\|^2 \\ \|x_{n+1} - x_n\|^2 \leq \alpha \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} \\ + \beta \{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\} + \gamma \|x_n - x_{n-1}\|^2 + \delta \{\|x_{n-1} - x_n\|^2 \\ + \|x_n - x_{n+1}\|^2\}$$

$$\{1 - 2\alpha - \beta - \delta\} \|x_{n+1} - x_n\|^2 \leq \{2\alpha + 2\beta + \gamma + \delta\} \|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq \frac{2\alpha + 2\beta + \gamma + \delta}{1 - 2\alpha - \beta - \delta} \|x_n - x_{n-1}\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq k \|x_n - x_{n-1}\|^2$$

Where $k = \frac{2\alpha + 2\beta + \gamma + \delta}{1 - 2\alpha - \beta - \delta}$. Continuing in the same way, we have

$$\|x_{n+1} - x_n\|^2 \leq k^n \|x_1 - x_0\|^2$$

Where $n = 1, 2, 3, \dots$, For any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\|^2 \leq \|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_{n+2}\|^2 + \dots + \|x_{m-1} - x_m\|^2$$

$$\leq k^n \|x_1 - x_0\|^2 + k^{n+1} \|x_1 - x_0\|^2 + \dots + k^{m-1} \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq k^n (1 + k + \dots + k^{m-n-1}) \|x_1 - x_0\|^2$$

$$\|x_n - x_m\|^2 \leq \frac{k^n}{1 - k} \|x_1 - x_0\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

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$$\leq \|u - x_n\|^2 + \alpha \frac{\|u - Tu\|^2 + \|x_{n-1} - Tx_{n-1}\|^2 + \|u - Tx_{n-1}\|^2}{1 + \|x_{n-1} - Tx_{n-1}\|^2 \|x_{n-1} - Tu\|^2} + \beta \{\|u - Tu\|^2 + \|u - Tx_{n-1}\|^2\} \\ + \gamma \|x_{n-1} - u\|^2 + \delta \max\{\|x_{n-1} - Tu\|^2, \|u - Tx_{n-1}\|^2\} + 2\operatorname{Re}\langle u - x_n, x_n - Tu \rangle$$

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$$\|v - u\|^2 \leq (\alpha + \beta + \gamma + \delta) \|v - u\|^2$$

Which is a contradiction; for $\alpha + \beta + \gamma + \delta < 1$. Thus u is a unique fixed point of T in X .

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