



Study of the Navier Stokes equations by SPH using the parametric kernel correction method

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Abstract: In this work we present a numerical method based on a corrected SPH approximation for solving the Navier-stokes equations in velocity pressure formulation that govern the flow of a viscous, incompressible and two-dimensional fluid in a top wall driven square cavity ..

As it is known, in the SPH method, the approximation of a function as well as the differential operators applied to a scalar or vectorial function are approached by choosing one of the kernel functions that exist. In this case, the spatial derivative of a function boils down to that of the kernel. But each kernel function has advantages and disadvantages and may agree to approach one of the terms of the Navier Stokes equations and not others. This work is based on this observation and allows thanks to the right choice of the kernel function for each term of the equations, and thanks to the correction of the pressure and viscosity part with renormalization, to eliminate the main disadvantages of the classic SPH method.

The Navier-stokes equations are formulated in terms of velocity pressure in a two-dimensional space. The temporal discretization of the Navier Stokes equations is done by a second-order Range_Kutta schema. Numerical results are presented, analyzed, and faced with other numerical results found by other digital methods (comparison with the finite difference method).

Keywords: Renormalization, Range_Kutta, The SPH method, Mesh free, Navier-stokes equations

I. Introduction

The smoothed-particle hydrodynamics (SPH) method gives the numerical solution of the equations of fluid dynamics by replacing the fluid through a set of particles. This is mesh free method based on a Lagrangian formulation. It was developed in 1977 by Gingold and Monaghan [1] and independently the same year by Lucy [2]. The motivation for the creation of this method came from the need to solve the complex problems that the methods known as Eulerian, as the method of finite difference, the finite element method or finite volume method had problems to solve.

Several advantages are related to the use of the SPH method for problem solving in physics or applied mathematics. Its use is not reserved for a particular geometry of the computational domain; It is easily usable for complex dimension 2 or 3 geometries. The problems with free boundaries that are encountered in Astrophysics are simple and natural with the SPH method, which is not the case of the methods with mesh. Because the SPH method is a method without mesh or grid, it allows to deal much more easily, compared to other methods, with problems which involve large deformations encountered in problems of explosion, high speed impact and penetration [3].

With the SPH method, it is easy to allow the borders to move or deform. The modeling of the interaction of several phases of a fluid bounded by a free surface is also made more easily. Thus, the interface problems can be easier to deal with the SPH method than with the finite differences method. In addition, for example for problems whose one of the unknowns (or the unknowns) has (have) a strong gradient, or the variations of the unknowns are concentrated in only some areas of the field of study, it is convenient to concentrate calculations in these regions. The choice of the SPH method is ideal for this type of problems, because it allows gaining in memory and computation time. We encounter this type of problems in fluid mechanics, thermal, nano and bioengineering. Finally, it is easier to implement digitally the SPH method and the extension in three dimensions is much easier than the methods with mesh (Gu. 2005) [4].

A. Principle and mathematical Formulation of the SPH method

The SPH method is based on three essential steps which will be presented in the following paragraphs:

- The representation of a function by its integral with the definition of the interpolation kernel ('Kernel approximation').

- The approximation particulate with the passage of the full form to a discrete sum of contributions of the particles present in the sphere of influence.
- The application of the formalism to the equations of conservation for the desired application [6].

B. Representation of a function by its integral

The concept of the representation of a function by its integral used in SPH is based on equality if the function f is defined and continuous on Ω

$$f(x) = \int_{\Omega} f(x') \delta(x - x') dx'$$

Where f is a function of the vector position ($\vec{x}$) and $\delta(x-x')$ the Dirac function defined as follows: $\delta(x - x') = \begin{cases} 1 & x=x' \\ 0 & x \neq x' \end{cases}$

By replacing the Dirac function δ by the interpolation kernel $W(x - x', h)$, We obtain the interpolating function $f(x)$ on Ω noted $\langle f(x) \rangle$ full. It is then written in the form $\langle f(x) \rangle$:

$$\langle f(x) \rangle = \int_{\Omega} f(x') W(x - x', h) dx' \quad (*)$$

Where W should check on Ω the requirement of unity, the convergence to the Dirac function when the smoothing length h tends to 0 and support must be compact

$$\int_{\Omega} W(x - x', h) dx' = 1 \quad (1)$$

$$\lim_{h \rightarrow 0} W(x - x', h) = \delta(x - x') \quad (2)$$

$$W(x - x', h) = 0 \text{ if } |x - x'| > kh \quad (3)$$

Where k is a constant of the interpolation function [7].

If the interpolation kernel is differentiable, one can construct the interpolating differential of function f noted $G[f]$.

$$G[f] = \int_{\Omega} \nabla f(x') \cdot W(x - x', h) dx' = - \int_{\Omega} f(x') \cdot \nabla W(x - x', h) dx' \quad (4)$$

We immediately notice that the gradient operator is transmitted to the interpolation function and the estimation of the derivative of a function can then be determined from the values of the function and the derivative of the kernel.

If the nucleus of interpolation is twice differentiable, one can construct the interpolating the Laplacien of the function f noted $L[f]$

$$L[f] = \int_{\Omega} \Delta f(x') \cdot W(x - x', h) dx' = \int_{\Omega} f(x') \cdot \Delta W(x - x', h) dx' \quad (5)$$

C. Choosing the kernel

The kernel chosen to approximate the Dirac function can be for example a Gaussian function, or spline. Compliance with the conditions (1) and (2) helping to build these functions. In practice, the kernel support is not the Ω of calculation field but support compact $\mathcal{D}$ (i.e. above a size equivalent to n times the value of the kernel smoothing length is zero). The most commonly used nuclei have a support of radius $2h$ or $3h$. This will directly affect the number of neighbors and therefore the accuracy of the approximation. In the end, we propose to use the following nuclei (or $q = |r|/h$):

The cubic spline proposed by Monaghan [2], on a size $2h$ support:

$$W(r, h) = C_c \begin{cases} \frac{2}{3} - q^2 + \frac{1}{2} q^3 & \text{si } 0 \leq q \leq 1 \\ \frac{1}{6} (2 - q)^3 & \text{si } 1 \leq q \leq 2 \\ 0 & \text{sinon} \end{cases} \quad (6)$$

Where the normalizing constant C allows to check the condition (1) and is worth $\frac{1}{h}$, $\frac{15}{7\pi h^2}$ and $3/(2\pi h^3)$, respectively in 1, 2 and 3 dimensions.

It is important to note that this kernel function does not depend as the distance between one point i and its neighbour j . Support is therefore spherical and the vicinity of a ball is actually composed of all adjacent balls to which $r_{ij} < 2h$.

The cubic spline proposed by [6], an h size support

$$W(r, h) = C_c \begin{cases} \frac{1}{2}q + q^2 - \frac{1}{2}q^3 - 1 & \text{si } 0 < q \leq 1 \\ 0 & \text{si } 1 \leq q \end{cases} \quad (7)$$

$$\lim_{q \rightarrow 0} W(r, h) = 0$$

Where the normalizing constant C allows to check the condition (1) and is worth $\frac{5}{h}$, $\frac{45}{7\pi h^2}$ and $\frac{15}{2\pi h^3}$ respectively in 1, 2 and 3 dimensions.

II. . The flow in a wall driven cavity

We will focus on solving the classical problem of an unsteady plane flow of viscous fluid incompressible in a rectangular wall hole upper driven. This flow is governed by the Navier Stokes equations and the equation of continuity in addition to the requirements to the limitations specified in the following figure:

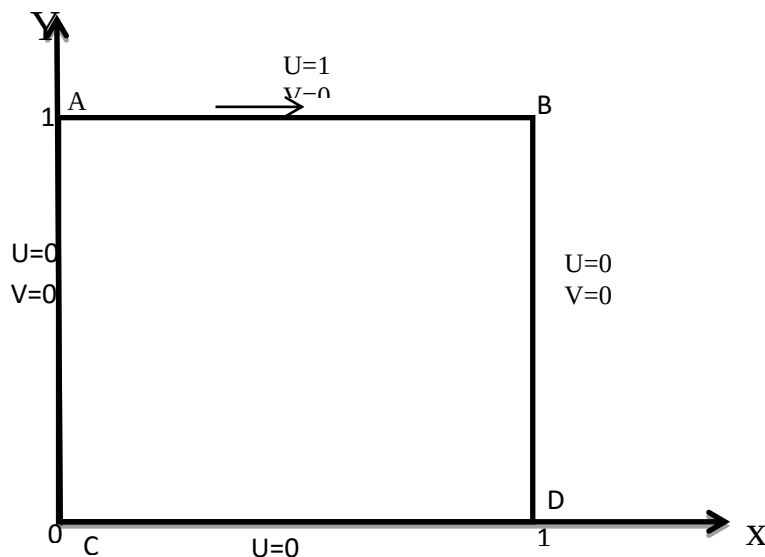


Figure 1: Boundary condition

The flow is generated under the influence of viscosity by the movement of the upper wall, which is animated by one uniform translation velocity, the other walls being immobile.

In the case of isothermal flows, the Navier-Stokes equations are easily obtained by the application of classical principles of conservation: conservation of mass, and momentum.

These equations are written in the following vector form:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} = -\nabla \cdot (\rho \mathbf{u}) + \mathbf{u} \cdot \nabla \rho \quad (8)$$

The continuity equation is given by :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (9)$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla P + \mu \Delta \mathbf{u} + \mathbf{f}_b \quad (10)$$

P : pressure

$\mathbf{u}$: velocity vector

$\mathbf{f}$: volumic force

μ : viscosity coefficient

To close the problem need to add initial conditions as well as the boundary conditions earlier given.

A. Formulation SPH

We apply the SPH formulation on the equation of continuity:

$$\frac{D\rho_i}{Dt} = -\nabla_i \cdot (\rho \mathbf{u}) + \mathbf{u}_i \cdot \nabla_i (\rho) \quad (11)$$

$$\frac{D\rho_i}{Dt} = -\sum_j \frac{m_j}{\rho_j} (\rho_j \mathbf{u}_j) \cdot \nabla_i \mathbf{W}_{ij} + \mathbf{u}_i \cdot \sum_j \frac{m_j}{\rho_j} \rho_j \nabla_i \mathbf{W}_{ij} \quad (12)$$

$$\frac{D\rho_i}{Dt} = \sum_j m_j (\mathbf{u}_i - \mathbf{u}_j) \cdot \nabla_i \mathbf{W}_{ij} \quad (13)$$

In this approach, the density of a particle changes only when it is in motion relative to other particles. Note that the density of a particle may also be calculated from the mass of neighboring particles with the equation

$$\rho_i = \sum_j m_j (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{W}_{ij} \quad (14)$$

But then a particle near the free surface do not miss the contribution of particulate matter above the free surface and the density becomes too weak. The calculation of the time derivative of the density does not lead to densities that are too low on the free surface.

The momentum conservation equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mu \Delta^2 \mathbf{u} + \mathbf{f}_b \quad (**)$$

The kernel functions are not necessarily the same for each calculation. Thus, the three different kernels will be used for each property. They are chosen by their simplicity, velocity of calculation and stability.

For the calculation of the density uses W_{poly6} While the pressure and viscosity calculations use W_{spiky} and, W_{visc} respectively.

$$W_{poly6}(\tilde{r}, h) = \frac{4}{\pi h^8} \begin{cases} (h^2 - r^2)^3 & r \leq h \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

$$W_{spiky}(\tilde{r}, h) = \frac{10}{\pi h^5} \begin{cases} (h-r)^3, & r \leq h \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

$$W_{visc}(\tilde{r}, h) = \frac{20}{\pi h^5} \begin{cases} -\frac{r^3}{2h^3} + \frac{r^2}{h^2} + \frac{h}{2r} - 1, & r \leq h \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

Where $r = \|\tilde{r}\|$.

The coefficients in front of each kernel are necessary to ensure the ownership of the unit. They are valid only in the 2D and are therefore valid for our application square cavity.

Accuracy of the algorithm depends heavily on smoothing kernels. For our purposes, we used the following core:

$$W_{poly6}(\tilde{r}, h) = \frac{4}{\pi h^8} \begin{cases} (h^2 - r^2)^3 & r \leq h \\ 0 & \text{otherwise} \end{cases}$$

To calculate the density of a particle from the neighboring particle mass

$$\rho_i = \sum_j m_j (\mathbf{u}_i - \mathbf{u}_j) \cdot W_{poly}(\tilde{r}, h) \quad (18)$$

The core functions used by Müller and his collaborators are also used in (Müller et al. 2003). Instead of an equation described by rule SPH a modified solution is used for the part of pressure because it guarantees the symmetry:

$$-\nabla P_i = -\sum_j m_j \frac{P_i - P_j}{2\rho_j} \nabla W_{spiky}(r_i - r_j, h) \quad (19)$$

Fluids are simulated using the SPH are compressible, as a rigid State equation is used. This is to avoid solving Poisson's equation for incompressible fluids at each time step. The equation of State is used in this work is derived from Batchelor (1967), and which is:

$$P = B \left[\left(\frac{\rho}{\rho_0} \right)^\gamma - 1 \right] \quad (20)$$

With pressure p , $\gamma = 7$, $\rho_0 = 1000 \frac{kg}{m^3}$: initial density of the fluid

$$B = \frac{c^2 \rho_0}{\gamma} \quad (21)$$

With c speed of sound

The formula of SPH, at the end of viscosity also gives an asymmetric term [7]:

$$\mu \Delta u_i = \mu \sum_j \frac{m_j}{\rho_{moyij}} \Delta w_{visco} (r_i - r_j, h) \quad (22)$$

Where $\rho_{moyij} = \frac{\rho_i \rho_j}{2(\rho_i + \rho_j)}$.

Finally, for the acceleration has a particle i we

$$\frac{du_i}{dt} = \frac{1}{\rho_i} (f_1 + f_2) \quad (23)$$

With $f_1 = -\sum_j m_j \frac{P_i - P_j}{2\rho_j} \nabla w_{spiky} (r_i - r_j, h)$ and $f_2 = \mu \sum_j \frac{m_j}{\rho_{moyij}} \Delta w_{visco} (r_i - r_j, h)$

B. Integration temps

We used a simple time of Euler integration, which is first order. The accuracy of position and velocity, can be written as follows:

$$\begin{aligned} u_i(t + \Delta t) &= u_i(t) + \Delta t \frac{du_i}{dt} \\ x_i(t + \Delta t) &= x_i(t) + \Delta t u_i(t + \Delta t) \end{aligned} \quad (24)$$

C. Boundary condition

The boundary conditions is a very important task for the SPH. Succeed a simulation returns in large part to properly model the boundary conditions. There are alternatives that can be adapted to SPH method thereby concerning modeling conditions to the spatial limits, velocity or inter-reaction. Based on the nature of the SPH, getting the value of a physical quantity at a given point, necessarily involves a smoothing over the vicinity of the point. Therefore, a problem emerges for points in the boundaries of the domain as well as locations that do not have enough space to smooth.

In this work we are interested in the method of ghost particle it comes around the field by a series of particles that are called ghost particle. This designation comes from the fact that these particles are nothing other than to retrieve a neighborhood of sufficient smoothing

The number and spatial distribution of these series of particles are determined to keep the kernel smoothing lengths ($2h$ for the case of a Spline type core and $4h$ in the case of the Gaussian). For example, to get the condition $u \cdot n = 0$ (n being the normal at the border, u the velocity vector), it is sufficient to assign to the first row of particles ghosts a velocity vector which verifies the following condition:

$$\frac{1}{2} (u_k + u_i) \cdot n = 0 \quad (25)$$

k being the last real particle and i being the first ghost particghostle. n is normal on the border. The half vector sum is required to retrieve the velocity in the middle of the two points mentioned above, which corresponds exactly to the boundary of the domain. Given that the normal vector n is known as well as the velocity u_k to the particle k , it is sufficient to solve the linear system generated by the equation. In the simple case where the border is right, the problem is simplified and confined to the opposite of the value of the component according to n of the velocity of the particle k to the phantom i particle. The ghost particle method has the advantage of being simple and effective for conventional geometries. However, it could be difficult to implement for complicated geometries. In addition, the addition of ghost particles represents a memory burden, especially where their number would be not negligible [9].

III. The parametric kernel correction method

A. Renormalisation

We now present some guidelines to improve SPH interpolation operators. Recall that presented continuous interpolation was analyzed in the case where the points are far enough of the field edges to prevent integration

support to cross the border (free surface or solid wall) [10]. If the proposed particle is near the wall, then the condition (1) is not verified, and therefore the estimation (*) shall be amended as follows:

$$\langle A(r,t) \rangle = A(r,t) \int_{\Omega} w(r) d^n r' + O(h^2) \quad (26)$$

This estimate is so close to the interpolated function as long as an interpolation operator continues to be defined as different from (*), noted $[A]^\lambda$

$$A(r,t) \square [A]^\lambda(r,t) = \frac{\langle A(r,t) \rangle}{\lambda(r)} \quad (27)$$

$$\lambda(r) = \int w_h(|r-r'|) d^n r' \quad (28)$$

With

Far from the border of the area, $\lambda(r) = 1$ (*); where $\langle A \rangle = [A]^\lambda$ while near the border, $\lambda(r)$ is less than 1. It should be noted that the definition (26) is formally identical to the classic Tween (*), under the condition that the kernel is corrected by

$$w_h(r,r') = \frac{w_h(|r-r'|)}{\lambda(r)} \succ w_h(|r-r'|) \quad (29)$$

The disadvantage of this formalism is seen as the corrected kernel w_h is no longer a simple function of the standard $\|r - r'\|$. As a result, most estimates of the highest precision announced above will be modified. The operation of discrete interpolation can be amended in the same manner. By analogy with (26), the interpolating discrete $\langle A \rangle^\lambda$ as defined by:

$$A(r_i,t) \approx \langle A(r_i,t) \rangle^\lambda = \frac{\langle A(r_i,t) \rangle}{\lambda_i} \quad (30)$$

On note $\lambda(r_i)$ par λ_i

It should be noted that λ_i is close to 1 for particles inside the area, but it cannot be equal to 1 for those close to the border. This new kernel is advantageous, because it satisfies the condition of normalization:

$$\begin{aligned} \langle 1 \rangle^\lambda &= \frac{1}{\lambda_i} \sum_j v_j w_{ij} \\ &= 1 \end{aligned} \quad (31)$$

With $v_j = \frac{m_j}{\rho_j}$

B. Correction of the gradient operator

The gradient operators can be defined (26). It is worth noting that the derivative of λ_a in regard to the r_a is written as follows:

$$\begin{aligned} \lambda'_a &= \frac{\partial \lambda_a}{\partial r_a} \\ &= \sum_b v_a \nabla w_{ab} \\ &= G_a [1] \end{aligned} \quad (32)$$

Where G_a is the standard discrete gradient as above defined operator

$$\lambda'_a \square \int_{\Sigma_a} w_h(|r'-r_a|) n(r') d^{n-1} \Gamma(r') \quad (33)$$

(Here, n is the vector of the unit towards the Interior, locally normal to the wall). We can now define a fixed gradient operator, as G^λ :

$$\begin{aligned} \frac{\lambda}{a} G[A_b] &= \frac{\partial \langle A_b \rangle^\lambda}{\partial r_a} \\ &= \frac{1}{\lambda_a} \sum_b v_b \nabla w_{ab} - \frac{1}{\lambda_a^2} \langle A \rangle G_a[1] \end{aligned} \quad (34)$$

The term $\frac{-1}{\lambda_a^2} \langle A_a \rangle G_a[1]$ expresses a so-called force generation force gradient, it will be noted by

$\frac{corr}{F_a}$
 G

Applying this operator to a constant function equal to 1, using (33), we find:

$$\begin{aligned} \frac{\lambda}{a} G[1] &= \frac{1}{\lambda_a} \sum_b v_b \nabla w_{ab} - \frac{1}{\lambda_a^2} \langle 1 \rangle G_a[1] \\ &= 0 \end{aligned} \quad (35)$$

This procedure could obviously be invoked successfully to correct the divergence operator. Indeed:

$$\begin{aligned} \frac{\lambda}{a} D[A_b] &= \frac{\partial}{\partial r_a} \cdot \langle A_b \rangle \\ &= \frac{1}{\lambda_a} D_a[A_b] - \frac{1}{\lambda_a^2} \langle A \rangle D_a[1] \\ \frac{\lambda}{a} D[1] &= 0 \end{aligned} \quad (36)$$

C. Correction of the Laplace operator

The Laplacian operators can be defined (33). It is worth noting that the derivative of λ_a in regard to r_a is written as follows:

$$\begin{aligned} \frac{\lambda}{a} L[A_b] &= \frac{\partial^2 \langle A_b \rangle^\lambda}{\partial r_a^2} \\ &= \frac{1}{\lambda_a} \sum_b v_b A_b \Delta w_{ab} + \frac{2}{\lambda_a^3} \langle A_a \rangle (G_a[1])^2 \\ &\quad - \frac{1}{\lambda_a^2} (G_a[A_a] \cdot G_a[1] + \langle A_a \rangle L[1] + G_a[1] \cdot G_a[1]) \end{aligned} \quad (37)$$

Where $L_a[1] = \sum_b v_b \Delta w_{ab}$ and $G_a[A_a] = \sum_b v_b A_b \nabla w_{ab}$.

The term $\frac{2}{\lambda_a^3} \langle A_a \rangle (G_a[1])^2 - \frac{1}{\lambda_a^2} (G_a[A_a] \cdot G_a[1] + \langle A_a \rangle L[1] + G_a[1] \cdot G_a[1])$ expresses a so called force generation

force Laplacien, it will be noted by $\frac{corr}{F_a}$
 L .

For a constant function, this operator is null, indeed:

$$\begin{aligned} \frac{\lambda}{a} L[1] &= \frac{1}{\lambda_a} \sum_b v_b \Delta w_{ab} + \frac{2}{\lambda_a^3} \langle 1 \rangle (G_a[1])^2 \\ &\quad - \frac{1}{\lambda_a^2} (G_a[1] \cdot G_a[1] + \langle 1 \rangle L[1] + G_a[1] \cdot G_a[1]) \\ &= 0 \end{aligned} \quad (38)$$

D. Correction of the Navier Stokes equation

The new set of equations takes advantage of the properties of renormalization of the global core.

$$\rho \frac{Du}{Dt} = \underbrace{-\frac{1}{\lambda} \nabla P}_{\text{pression}} + \underbrace{\mu \frac{1}{\lambda} \Delta u}_{\text{viscosite}} + \underbrace{F}_{\text{intérieur}} \quad (39)$$

$$\text{Where } F_{\text{intérieure}} = F_{\text{corr}} + F_{\text{L}}$$

We see that the only differences with the original Navier-Stokes equations are the use of factors λ and addition of the strength of gradient and Laplacian of. This strength comes from the correction of the kernel for the border near particles.

IV. Numerical results and Discussion

The geometry of the problem is a side square cavity $L = 1$. The edges of the cavity are the walls, except the top edge for which is prescribed a uniform tangential velocity ($u_x = 1$ and $u_y = 0$). Several calculations have been performed in the present study, with numbers of Reynolds's 1, 10, 100 and 1000 respectively.

In this case we take different numbers of particles depending on x and next y $n_x = n_y = 10$, $n_x = n_y = 20$, $n_x = n_y = 30$ and $n_x = n_y = 40$; all these calculations were performed with the same time step $dt = 5.10^{-5}$ after a number of time steps $ndt = 3000$. The dimension on the side of the square field is $L = 10^{-3}m$

In the beginning, a total of 100 real particles (circular dots) are placed in the square box, $n_x = n_y = 10$

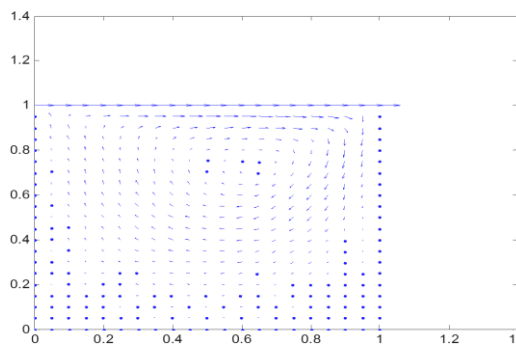


Figure 2 number of particle 20/20 Re=1000

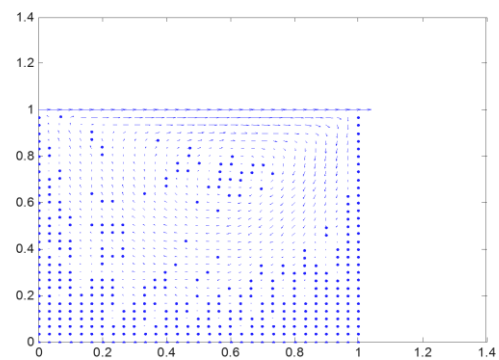


Figure 5 number of particle 30/30 Re=1000

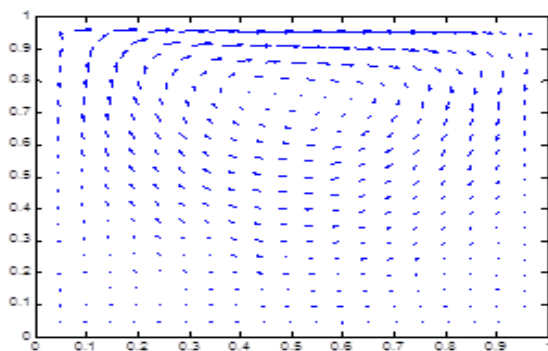


Figure 3 number of particle 20/20 Re=1000[R. Liu method]

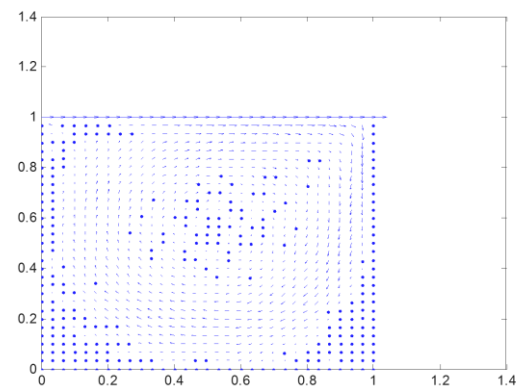


Figure 6 number of particle 40/40 Re=1000

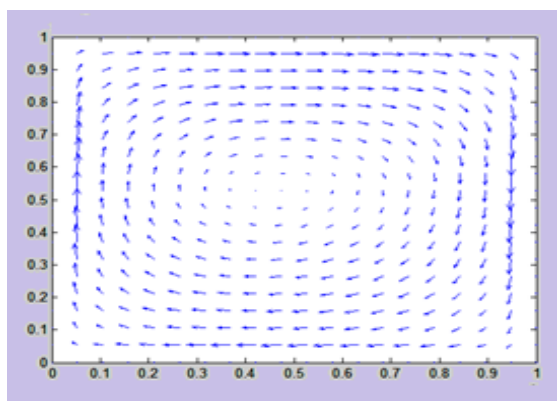


Figure 4 number of particle 20/20 Re=100(the finite difference method)

These figs show the distribution of velocities of the particles for different number of Reynolds. One can clearly observe the flow recirculation

V. Conclusion

We have presented in this work an analysis of Navier-stokes equations using an SPH formulation. The main disadvantages of the classic SPH method have been eliminated with a good choice of the kernel function for each part of the equation as well as the correction of pressure and viscosity part with the renormalization for greater accuracy and consistency conservation. We found that the method of parametric correcting λ of consistency gives more suitable results.

The conducted numerical tests lead to the conclusion that our approach has allowed to reproduce for different values of the Reynolds number G-LIU results [(GR Liu nd)] as well as those given in other studies with the finite difference method of H2-H4 order

We are currently working on the extension of this method for natural convection in dimension 2 and

VI. References

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