



Applying Hedges to the first component of Z-numbers

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Abstract: Zadeh defined a Z-number associated with an uncertain variable X as an ordered pair of fuzzy numbers (A, B) . Computation with Z-numbers is a topic which is both interesting as well as useful. In this paper we study the problem of applying hedges to the first component of Z-numbers.

Keywords: Z-numbers, Z-valuation, restriction, fuzzy event probability, hedges, modifier.

I. Introduction

Zadeh introduced the concept of Z-numbers in 2011[5]. Zadeh defined a Z-number associated with an uncertain variable X as an ordered pair of fuzzy numbers (A, B) . The first component is the restriction on the values which X can take while the second component is the measure of certainty of the information. Z-numbers are very useful in modeling verbal arguments. Computation with Z-numbers is a topic which is both interesting as well as useful. Yagers [4] and Shankar K Pal [3] have initiated research in this topic. However a lot remains to be done.

In this paper we study the problem of applying a hedge to the first component of Z-number. More precisely, if the Z-valuation " X is (A, B) " is known the aim is to compute the Z-valuation " X is (C, D) " where C is got by applying an hedge to A .

II. Preliminaries

Definition 1: Fuzzy Set

A fuzzy set A defined on a universe X may be given as:

$A = \{(x, \mu_A(x)) / x \in X\}$ where $\mu_A: X \rightarrow [0, 1]$ is the membership function of A . The membership value $\mu_A(x)$ describes the degree of belonging of $x \in X$ in A [7].

Definition 2: Restrictions

A restriction may be viewed as generalized constraint. Suppose A is a fuzzy set and X a variable. The statement $R(X) : X$ is A is referred to as a possibilistic restriction ([10],[8]). Here A plays the role of possibility distribution of X .

The statement " $R(X) : X$ is A " is to be understood as "The Possibility ($X=u$) is $\mu_A(u)$ " where μ_A is the membership function of A and u is a generic value of X .

When X is a random variable, the probability distribution of X plays the role of probabilistic restriction on X ([10],[2]). A probabilistic restriction is expressed as:

$R(X) : X$ is f , where f is the p.d.f. of X .

That statement " $R(X) : X$ is f " is to be understood as " $\text{Prob}(u \leq X \leq u+du) = f(u)du$ "

Zadeh [9] has given the following definition for the probability of a fuzzy event:

Definition 3 : Fuzzy event probability

Let X be a random variable taking real values and A a fuzzy set defined on the real line. The fuzzy event probability of " X is A " is

$$\text{FEP}(x \in A) = \int \mu_A(u)p(u)du$$

Definition 4: Z-number

Zadeh [5] defines Z-number as follows:

A Z-number is an ordered pair of fuzzy numbers $Z = (A, B)$, associated with a real-valued uncertain variable X , with the first component A , a restriction on the values which X can take and the second component B , a measure of reliability of the first component.

Note: Though Zadeh has defined a Z-number to be an ordered pair of fuzzy numbers (A, B) the components A and B may not be fuzzy numbers in the strict technical sense. Rather A should be considered to be a fuzzy subset defined on the real line and B should be considered to be a fuzzy set defined on the interval $[0, 1]$.

Definition 5: Z-valuation

A Z-valuation is an ordered triple (X, A, B) where A and B are fuzzy numbers[5]. A Z-valuation is equivalent to an assignment statement " X is (A, B) ", where X is an uncertain variable. A is a restriction on the values which X can take and B is referred to as certainty([8],[6]).

Z-valuation (X, A, B) may be viewed as a restriction on X defined by

$FEP(X \text{ is } A) \text{ is } B$

More explicitly

Possibility($FEP(x \in A) = u$) = $\mu_B(u)$

Definition 6: Z-restriction

A Z-restriction is expressed as

$R(X) : X \text{ is } Z$

where Z is a combination of possibilistic and probabilistic restriction defined as([10],[8])

$Z : EEP(X \text{ is } A) \text{ is } B$ in which A and B are fuzzy numbers

Equivalently, a Z-valuation (X, A, B) is a Z-restriction on X .

$(X, A, B) \rightarrow X \text{ is } (A, B)$

Definition 7: Linguistic Hedge

Linguistic hedges ([12],[1]) (or simply hedges) are special linguistic terms by which other linguistic terms are modified. Linguistic terms such as very, more or less, fairly, or extremely are examples of hedges. They can be used for modifying fuzzy predicates, fuzzy truth values, and fuzzy probabilities.

A linguistic hedge, H , may be applied to a fuzzy set by using an unary operation, h defined on the unit interval $[0, 1]$.

An unary operation, which is an increasing bijection, representing a linguistic hedge is called modifier.

If $\mu_A(u)$ is the membership function of a fuzzy set A then a linguistic hedge H can be applied to A by using the appropriate modifier h and defining

$$\mu_{HA}(u) = h(\mu_A(u))$$

Example

For example, the hedge 'very' is often interpreted as the unary operation $h(a) = a^2$ while the hedge 'fairly' is interpreted as $h(a) = \sqrt{a}$ ($a \in [0, 1]$).

Note

If $h(a) < a$ for all $a \in [0, 1]$, the modifier is called strong; if $h(a) > a$ for all $a \in [0, 1]$, the modifier is called weak. The modifier $h(a) = a$ for all $a \in [0, 1]$, is called an identity modifier[1].

Every modifier h satisfies the following conditions:

1. $h(0) = 0$ and $h(1) = 1$;
2. h is a continuous function;
3. if h is strong, then h^{-1} is weak and vice versa;
4. given another modifier g , compositions of g with h and h with g are also modifiers and, moreover, if both h and g are strong(weak), then so are the compositions.

Extension Principle

Suppose the possibility distribution of X is known. To compute the possibility distribution of $Y = f(X)$ we have to use the extension principle.

A general version of the extension principle was described in[11]. In this version, we have

$$Y = f(X)$$

$$R(X): g(X) \text{ is } A \text{ (constraint on } u \text{ is } \mu_A(g(u)))$$

$$R(Y): \mu_Y(v) = \sup_u \mu_A(g(u))$$

subject to

$$v = f(u)$$

For a function with two arguments, the extension principle reads:

$$Z = f(X, Y)$$

$$R(X): g(X) \text{ is } A \text{ (constraint on } v \text{ is } \mu_A(g(u)))$$

$$R(Y): h(Y) \text{ is } B \text{ (constraint on } v \text{ is } \mu_B(h(u)))$$

$$R(Z): \mu_Z(w) = \sup_{u,v} (\mu_X(g(u)) \wedge \mu_Y(h(v))), \wedge = \min$$

subject to

$$w = f(u, v)$$

III. Computation of Z-valuation involving hedges

Suppose " X is (A, B) " is a known Z-valuation. That is it is given that ' $FEP[X \text{ is } A]$ is B '. So the membership functions μ_A and μ_B are known.

$$\text{FEP}(X \in A) = \int \mu_A(u)p(u)du \text{ and } \mu_B(v) = \int \mu_A(u)p(u)du = \mu_B(v) \text{ is known.}$$

Let C be a fuzzy set obtained by applying a hedge to A. So we have

$$\mu_C(u) = h(\mu_A(u))$$

where h is the appropriate modifier.

Let $w = \text{FEP}(X \in C) = \int \mu_C(u)p(u)du$. The problem is to compute $\mu_D(w)$.

By extension principle,

$$\mu_D(w) = \sup_p \mu_B(v)$$

subject to $v = \int \mu_A(u)p(u)du$

and $w = \int \mu_C(u)p(u)du$.

Here the supremum is required to be taken over the set of all possible probability density functions p. In other words for any value of w, to compute $\mu_D(w)$, we have to solve a variational problem. This problem may be pretty much impossible to solve unless some information regarding the nature of the 'p' is available.

Example 1:

Suppose X isz (A, B) where

$$\mu_A(u) = \frac{10-u}{2}, \quad 8 \leq u \leq 10$$

$$\mu_B(v) = 0.3 \leq v \leq 0.35$$

We wish to compute the z-valuation X isz (C, D) for $C = H(A)$ where the hedge is represented by the modifier $h(a) = a^2$.

Assume that it is known that the underlying probability density function of X is known to be the uniform distribution given by

$$p(u) = \frac{1}{L}, \quad 10 - \frac{L}{2} \leq u \leq 10 + \frac{L}{2}$$

However the parameter L is not known.

$$\text{Then } \text{FEP}(X \in A) = v = \int \mu_A(u)p(u)du = \int_{10-\frac{L}{2}}^{10+\frac{L}{2}} \left(\frac{10-u}{2}\right) \frac{1}{L} du = \frac{L}{16}$$

$$\text{Also } \mu_C(u) = \left(\frac{10-u}{2}\right)^2, \quad 8 \leq u \leq 10$$

$$\begin{aligned} \therefore w &= \int_8^{10} \mu_C(u)p(u)du \\ &= \int_{10-\frac{L}{2}}^{10+\frac{L}{2}} \left(\frac{10-u}{2}\right)^2 \frac{1}{L} du = \frac{L^2}{96} \end{aligned}$$

By extension principle,

$$\mu_D(w) = \sup_p \mu_B(v) \tag{1}$$

$$\text{subject to } v = \int \mu_A(u)p(u)du = \frac{L}{16} \tag{2}$$

$$\text{and } w = \int \mu_C(u)p(u)du = \frac{L^2}{96} \tag{3}$$

In equation (1) the supremum is to be taken over all possible probability density functions 'p' subject to conditions (2) and (3). In essence the supremum is to be taken over all possible values of parameter L subject to (2) and (3). However using equations (2) and (3) we get

$$w = \frac{L^2}{96} = \frac{8}{3} \left(\frac{L}{16}\right)^2 = \frac{8}{3} v^2$$

$$\text{Thus } \mu_D(w) = 1 \quad \text{if} \quad 0.3 \leq v \leq 0.35$$

$$\text{that is } \mu_D(w) = 1 \quad \text{if} \quad 0.24 \leq w \leq 0.327$$

Example 2:

Let X isz (A, B) where

$$\mu_A(u) = \begin{cases} u, & 0 \leq u \leq 1 \\ 2-u, & 1 \leq u \leq 2 \end{cases}$$

$$\mu_B(v) = \begin{cases} 10(v-0.7), & 0.7 \leq v \leq 0.8 \\ 10(0.9-v), & 0.8 \leq v \leq 0.9 \end{cases}$$

We wish to compute the z-valuation X isz (C, D) for $C = H(A)$ where the hedge is represented by the modifier $h(a) = \sqrt{a}$.

Assume that it is known that the underlying probability density function of X is known to be the normal distribution given by

$$p(u) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{(x-1)^2}{2\sigma^2}}$$

That is $X \sim N(1, \sigma^2)$ where the parameter σ is not known.

$$\text{Then } v = \int \mu_A(u)p(u)du = \int_0^1 up(u)du + \int_1^2 (2-u)p(u)du$$

$$\begin{aligned}
 \text{and } w &= \int \mu_C(u) p(u) du \\
 &= \int_0^1 \sqrt{u} p(u) du + \int_1^2 \sqrt{(2-u)} p(u) du \\
 &= \int_0^1 (1 - (1-u))^{\frac{1}{2}} p(u) du + \int_1^2 (1 - (u-1))^{\frac{1}{2}} p(u) du \\
 &\cong \int_0^1 \frac{1}{2} p(u) du + \frac{1}{2} \int_0^1 u p(u) du + \int_1^2 \left(1 - \frac{u}{2}\right) p(u) du + \int_1^2 \frac{1}{2} p(u) du
 \end{aligned}$$

where the binomial theorem has been used to approximate the integrand.

$$\text{Thus } w \cong \frac{1}{2} \int_0^2 p(u) du + \frac{v}{2}$$

Since X is a normal random variable with mean 1 and variance σ^2

$$\begin{aligned}
 \int_0^2 p(u) du &= \text{Prob}(0 \leq x \leq 2) \\
 &= 2 \text{Prob}\left(0 \leq \frac{x-1}{\sigma} \leq \frac{1}{\sigma}\right)
 \end{aligned}$$

Here $z = \frac{x-1}{\sigma}$ is a standard normal variate.

$$\text{Therefore } \mu_D(w) = \sup_p \mu_B(v) \quad (4)$$

$$\text{Subject to } w = \text{Prob}\left(0 \leq z \leq \frac{1}{\sigma}\right) + \frac{v}{2} \quad \text{and} \quad (5)$$

$$v = \int_0^1 u p(u) du + \int_1^2 (2-u) p(u) du \quad (6)$$

The above equations are not so easy to solve. However an approximate solution can be obtained by using numerical methods.

IV. Conclusion

In this paper the procedure for applying a hedge to the first component of z-number is given. The procedure has been illustrated by appropriate examples. The second example shows that, even after simplifying assumptions are made regarding the underlying probability density function, computations involving z-numbers are complicated. So further study is required to see what other simplifying assumptions can be made in order to make the problem more tractable.

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