



On Volterra Type Integral Inequalities in Metric Spaces

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Abstract: In the present work we consider a class of abstract Volterra type integral equations of second kind. Sufficient conditions are obtained for existence and uniqueness of the solutions of this class integral equations. Some applications of the obtained results to the integral inequalities involving maxima are given.
Keywords: Volterra Type Integral Equations, Metric space, Integral Inequalities with maxima

I. Introduction

As an evergreen theme in the past few years the integral equations have proved to be of great use in several applied fields, such as population dynamics, automatic control theory, dynamics of nuclear reactors. The main application of the corresponding integral inequalities is that they provide explicit bounds of the unknown functions, which are a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of nonlinear differential and integral equations. The techniques of the proofs of the integral inequalities in general are based on the classical mathematical analysis and are leading up to virtuosity. The major problem is that these proofs significantly depend of the number of independent variables and the geometry of the domains of integration. A lot of information about the integral equations and inequalities can be found in the monographs [1] - [5] and the comprehensive list of references therein. Many results about differential equations and integral inequalities with maxima can be found in the fundamental monograph [6]. Note that the authors of [6] made substantial contributions in this field of investigations. The aim of this paper is to extend the approach introduced for the linear case in the work [7] and partially developed for the nonlinear one in [8]. The main idea introduced in [7] is to establish conditions under which the unique solution of the equation (2.1) is an upper bound of all solutions of the inequality (2.2). In view of the applications it is important to study also the cases when the domains of integration have different geometric forms.

The paper is organized as follows – in section 2 we introduce an abstract analogue of the Volterra equation and its corresponding inequality. Section 3 is devoted to the study of the Volterra type equation (2.1) introduced below. Section 4 includes the results concerning the corresponding integral inequality (2.2) and in section 5 are given some applications of the results obtained in the previous sections to the integral inequalities with maxima.

II. Preliminary Notes and Denotations

Let Ω be a complete metric space with metric ρ , $B_\Omega \subset 2^\Omega$ denotes the σ -algebra of the Borel subsets of Ω and let $\mu: B_\Omega \rightarrow [0, \infty]$ be a nontrivial, nonatomic σ -additive Borel measure. We will denote by $U(x, \varepsilon) = \{y \in \Omega \mid \rho(x, y) < \varepsilon\}$ the open balls with a center point $x \in \Omega$, and radius $\varepsilon > 0$. If G is an arbitrary subset of Ω then ∂G denotes the boundary of G and $U(G, \varepsilon)$ denotes the ε -neighborhood of G .

Let B be a real Banach space with norm $\|\cdot\|_B$, B^* be the Banach space of all linear bounded operators acting in B with the norm $\|\cdot\|_{B^*}$ and $V \subset B$ be a cone in B . Then we can introduce a partial order in B associated with the cone V , i.e. $u \geq v$ when $u - v \in V$. We shall write $u > v$ to indicate that $u - v \in V$, but $u \neq v$.

Denote by $C(G, B)$, where $G \subset \Omega$ is an arbitrary compact subset, the Banach space of all continuous maps $f: G \rightarrow B$ with norm $\|f\|_G = \sup_{y \in G} \|f(y)\|_B$.

Definition 2.1. [8] The point $x \in \Omega$ will be called essential for the set G , if for each $\varepsilon > 0$ we have $\mu(G \cap U(x, \varepsilon)) > 0$. Otherwise the point $x \in \Omega$ will be called nonessential.

We will denote by G^μ the set of all points $x \in G$, which are essential for G and by G^ν the set $G^\nu = G \setminus G^\mu$, i.e. G^ν includes all nonessential points belonging to G .

Definition 2.2. [9] The sets $G, H \in B_\Omega$, will be called μ -equivalent $G \sim^\mu H$, if $\mu(G \Delta H) = 0$, where $G \Delta H = (G \setminus H) \cup (H \setminus G)$ is the symmetric difference.

Definition 2.3. [8], [9] The set $G \in B_\Omega$, is called μ -dense, if each $x \in G$ is an essential point for the set G .

Definition 2.4. [9] We say that the set $G \subseteq \Omega$ is M -star if for every $x \in G$ the inclusion $M_x \subseteq G$ holds.

Remark 2.5. From Definition 2.1. it follows, that for arbitrary $G \in B_\Omega$ with $\mu(G) > 0$ every external or isolated point of G is nonessential and every internal point is essential for G . Then it follows if $\mu(G) = 0$ and $G \neq \emptyset$ then G not include essential points. Moreover if the point $x \in \Omega$ is nonessential for the set G , then there exists $\varepsilon_0 > 0$ that $\mu(G \cap U(x, \varepsilon_0)) = 0$.

Remark 2.6. It is simply to see that condition A3 implies that for each $x \in \Omega$ the set M_x is M -star set. Moreover the union and the intersection of an arbitrary family of M -star sets are M -star set. The sets $\text{Ker} M = \{x \in \Omega \mid \mu(M_x) = 0\}$ and $M_\Omega = \bigcup_{x \in \Omega} M_x$ are M -star sets too.

Following [7] we introduce the map $M : \Omega \rightarrow 2^\Omega$, which associates every point $x \in \Omega$ with a subset $M_x \subset 2^\Omega$.

We will say that the conditions (A) hold if for the map $M : \Omega \rightarrow 2^\Omega$ the following conditions are fulfilled:

A1. For every point $x \in \Omega$ the set M_x is compact.

A2. For each $\varepsilon > 0$ and every $x \in \Omega$, there exists $\delta > 0$ such that for each $y \in \Omega$ with $\rho(y, x) < \delta$ we have that $\mu(M_x \Delta M_y) < \varepsilon$.

A3. For every $x \in \Omega$, the inclusion $M_y \subseteq M_x$ holds for each $y \in M_x$.

A4. There exists $x_0 \in \Omega$ such that $\mu(M_{x_0}) = 0$.

Definition 2.7. Every map $M : \Omega \rightarrow 2^\Omega$ for which the conditions (A) hold will be called Volterra type (VT map). We say that a VT map M is μ -dense if the sets M_x are μ -dense for each $x \in \Omega$.

Denote by $\mathbf{M}$ the set of all VT maps and by $\mathbf{M}_0$ the set of all μ -dense VT maps. For each $M \in \mathbf{M}$ denote with $\tilde{M} = \{M_x \mid x \in \Omega\}$ the set of all sets which are values of the map M . We introduce partial order in the set $\mathbf{M}$ as follow: $M^* \prec M$ if we have that $\text{Ker} M^* \subseteq \text{Ker} M$.

Let $M, M^* \in \mathbf{M}$ be arbitrary maps and consider the equation

$$f(x) = p(x) + \int_{M_x^*} W(x, y) f(y) d\mu_y + \int_{M_x} Q(x, y, f(y)) d\mu_y \quad (2.1)$$

and the corresponding inequality

$$g(x) \leq p(x) + \int_{M_x^*} W(x, y) g(y) d\mu_y + \int_{M_x} Q(x, y, g(y)) d\mu_y, \quad (2.2)$$

where the operators $Q : \Omega \times \Omega \times B \rightarrow B$, $W : \Omega \times \Omega \rightarrow B^*$ and $f, g, p \in C(M_x, B)$, $x \in \Omega$.

Remark 2.8. If the operator Q is continuous in the set $\Omega \times M_\Omega \times B$ and the operator-valued function $W : \Omega \times \Omega \rightarrow B^*$ is continuous in $\Omega \times M_\Omega$, then the Bochner integrals in (2.1) and (2.2) exist on each $M_x^* \in \tilde{M}^*$ and $M_x \in \tilde{M}$. ([10], Chapter 3).

For each map $M \in \mathbf{M}$ we define its associated map M'' for each $x \in \Omega$ by the relation $M_x'' = (M_x)''$.

Remark 2.9. From Lemma 3.11 and Lemma 3.14 of [8] it follows, that for every map $M \in \mathbf{M}$ its associated map M'' is μ -dense, i.e. $M'' \in \mathbf{M}_0$ and for each $x \in \Omega$ we have that $\mu(M_x'' \Delta M_x) = 0$. Thus without loss of generality we can use for every $x \in \Omega$ as a domain of integration instead M_x its μ -equivalent domain M_x'' which is μ -dense.

III. Main Results

In our discussion below we will assume that for the operator $Q : \Omega \times \Omega \times B \rightarrow B$ and the operator-valued function $W : \Omega \times \Omega \rightarrow B^*$ are fulfilled some of the following conditions:

(S1) The operator $Q : \Omega \times \Omega \times B \rightarrow B$ is continuous in the set $\Omega \times M_\Omega \times B$ and the operator-valued function $W : \Omega \times \Omega \rightarrow B^*$ is continuous in $\Omega \times M_\Omega$.

(S2) For each $x \in \Omega$ and every $f \in C(M_x, B)$, there exist numbers $\delta(M_x, f) > 0$ and $L(M_x, f, \delta) > 0$ such that for every function $g \in C(M_x, B)$ for which $\|f - g\|_{M_x} < \delta$, the following inequality holds:

$$\sup_{(s, y) \in M_x \times M_x} \|Q(s, y, f(y)) - Q(s, y, g(y))\|_B \leq L(M_x, f, \delta) \|f - g\|_{M_x}.$$

(S3) For every $x \in \Omega$ and arbitrary $r > 0$, there exists a constant $L(M_x, r) > 0$, such that for every two functions $f, g \in \{f \in C(M_x, B) \mid \|f\|_{M_x} \leq r\}$ the following inequality holds:

$$\sup_{(s,y) \in M_x \times M_x} \|Q(s, y, f(y)) - Q(s, y, g(y))\|_B \leq L(M_x, r) \|f - g\|_{M_x}.$$

(S4) For each M -star continuum $\Omega_* \subset \Omega$ there exists a number $L(\Omega_*) > 0$, such that for every two functions $f_1, f_2 \in C(\Omega_*, B)$ and for each $x \in \Omega_*$ the following inequality holds

$$\|Q(s, y, f_1(y)) - Q(s, y, f_2(y))\|_B \leq L(\Omega_*) \|f_1(y) - f_2(y)\|_B.$$

Consider the operators K and W_* defined by the equations

$$Kf(x) = p(x) + \int_{M_x} Q(x, y, f(y)) d\mu_y \quad (3.1)$$

$$W_* f(x) = \int_{M_x^*} W(x, y) f(y) d\mu_y, \quad (3.2)$$

and the operator T defined by

$$Tf(x) = Kf(x) + W_* f(x) \quad (3.3)$$

where $f, p \in C(\Omega, B)$. As we mentioned above in Remark 2.8, if the condition S1 holds, then the Bochner integrals in (3.1) and (3.2) exist on each $M_x^* \in \tilde{M}^*$ and $M_x \in \tilde{M}$.

Remark 3.1. Obviously if for each M -star continuum $\Omega_* \subset \Omega$ the operator-valued function W is continuous in $\Omega_* \times \Omega_*$ then for it the conditions (S3), (S3) and (S4) hold.

Lemma 3.2. Let the following conditions be fulfilled:

1. The conditions (S1) and (S2) hold.
2. Ω is a connected metric space and $0 < \mu(\Omega) \leq \infty$.
3. The maps $M, M^* \in \mathbf{M}_0$ and the relation $M^* \prec M$ holds.

Then the operator T defined by the equality (3.3) maps $C(\Omega_*, B)$ into $C(\Omega_*, B)$ continuously for each $p \in C(\Omega_*, B)$.

Proof: The proof that the operator K defined by the equality (3.1) maps $C(\Omega_*, B)$ into $C(\Omega_*, B)$ continuously for each $p \in C(\Omega_*, B)$ is completely analogical of the proof of Lemma 4.1 in [8] and will be omitted.

Condition (S1) and condition 3 of Lemma 3.2 implies that there exists a constant $W_0(\Omega_*) > 0$ such that for each $x \in \Omega_*$ and $y \in M_x^*$ the following estimation holds

$$\|W_* f(x)\|_B \leq \int_{M_x^*} \|W(x, y)\|_B \|f(y)\|_B d\mu_y \leq W_0(\Omega_*) \mu(M_x^*) \|f\|_{M_x^*} \leq W_0(\Omega_*) \mu(\Omega^*) \|f\|_{M_x^*}, \quad (3.4)$$

where $f \in C(M_x^*, B)$ is arbitrary. Then (3.4) implies that the operator W_* defined by the equality (3.2) maps $C(\Omega_*, B)$ into $C(\Omega_*, B)$ continuously and therefore from (3.3) it follows that the operator T maps $C(\Omega_*, B)$ into $C(\Omega_*, B)$ continuously for each $p \in C(\Omega_*, B)$. $\square$

Let $x \in \Omega$ be an arbitrary point and $\Omega_* \subset \Omega$ is arbitrary M -star continuum. Then for every $f \in C(\Omega_*, B)$ there exist two points $s_*^m, s_*^M \in \Omega_*$ such that $\|f(s_*^m)\|_B = \min_{s \in \Omega_*} \|f(s)\|_B$ and $\|f(s_*^M)\|_B = \max_{s \in \Omega_*} \|f(s)\|_B$.

Definition 3.3. A continuous function $F: \Omega \rightarrow B$ is called an extension of the function $f \in C(\Omega_*, B)$ if $F(s) = f(s)$ for each $s \in \Omega_*$. If in addition for some extension $F(s)$ and each $s \in \Omega$ the inequalities $\|f(s_*^m)\|_B \leq \|F(s)\|_B \leq \|f(s_*^M)\|_B$ hold, then such an extension will be called middle extension.

Definition 3.4. We say that the equation (2.1) has a local solution in some M -star set $\Omega_* \subset \Omega$ for some $p \in C(\Omega_*, B)$ if there exist a point $x_p \in \Omega_*$ with $\mu(M_{x_p}) > 0$ and a function $f \in C(M_{x_p}, B)$ which satisfies the equation (2.1) for each $s \in M_{x_p}$. If $p, f \in C(\Omega_*, B)$ and f satisfies the equation (2.1) for each $x \in \Omega_*$ then we say that f is a solution of (2.1) in $\Omega_* \subset \Omega$.

Theorem 3.5. Let the following conditions be fulfilled:

1. The conditions (S1) and (S3) hold.
2. Conditions 2 and 3 of Lemma 3.2 hold.

Then for every $x \in \Omega$ such that $M_x \setminus \text{Ker} M \neq \emptyset$ and for every $p \in C(M_x^* \cup M_x, B)$ the equation (2.1) has at least one local solution in $M_x^* \cup M_x$.

Proof: Let $\text{Ker} M \subset \Omega$ (the case when $\text{Ker} M = \Omega$ is trivial). For each $x \in \Omega$, from Lemma 3.16 [8], it follows that $\text{Ker} M \cap M_x \neq \emptyset$, $\text{Ker} M^* \cap M_x^* \neq \emptyset$ and then condition 2 implies that $\text{Ker} M \cap M_x^* \neq \emptyset$ also. Theorem 3.18 [14] implies that there exist a point $y_0 \in \partial \text{Ker} M \cap M_x^*$ and a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq M_x^* \setminus \text{Ker} M^*$, such that $\lim_{n \rightarrow \infty} \rho(y_n, y_0) = 0$. Then for the sequence $\{\mu(M_{y_n}^*)\}_{n \in \mathbb{N}}$ we have that $\mu(M_{y_n}^*) > 0$ for each $n \in \mathbb{N}$ and if the set $\{\mu(M_{y_n}) \mid \mu(M_{y_n}^*) > 0\}_{n \in \mathbb{N}}$ is finite then the statement of Lemma 3.2 follows from Lemma 4.1 [8].

Let us consider the case when the set $\{\mu(M_{y_n}) \mid \mu(M_{y_n}^*) > 0\}_{n \in \mathbb{N}}$ is infinite. Then there exists a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}} \subseteq \{y_n\}_{n \in \mathbb{N}} \subseteq M_x^* \setminus \text{Ker} M^*$, such that $\mu(M_{y_{n_k}}) > 0$ for each $k \in \mathbb{N}$ and condition 2 implies that $\lim_{k \rightarrow \infty} \mu(M_{y_{n_k}}) = 0$.

Let us denote $A_x = M_x^* \cup M_x$, $A_k = M_{y_{n_k}}^* \cup M_{y_{n_k}}$, $k \in \mathbb{N}$, $P = \max_{s \in A_x} \|p(s)\|_B$, $Q_0 = \sup_{s, y \in A_x} \|Q(s, y, 0_B)\|_B$. For each $k \in \mathbb{N}$ let us define the operators T_k by the following equality

$$T_k f(s) = p(s) + \int_{M_x^*} W(s, y) f(y) d\mu_y + \int_{M_x} Q(s, y, f(y)) d\mu_y, \quad (3.5)$$

where $p, f \in C(A_x, B)$ and $s \in A_k$. For every function $f \in C(A_k, B)$ and each $k \in \mathbb{N}$ from Lemma 4.3 [8] it follows that there exists a middle extension, $F_k \in C(A_x, B)$ such that $\|F_k\|_{A_x} = \|f\|_{A_k}$.

Let $r > P$ be an arbitrary number and for every $r > P$ and each $k \in \mathbb{N}$ introduce the sets $U_k(r) = \{f \mid f \in C(A_k, B), \|f\|_{A_k} \leq r\}$, $\bar{U}_k(r) = \{F_k \mid f \in U_k(r)\}$ and $U_x(r) = \{F \mid F \in C(A_x, B), \|F\|_{A_x} \leq r\}$. Then for each $k \in \mathbb{N}$ we have that $\bar{U}_k(r) \subset U_x(r)$.

From Lemma 3.2 it follows that for each $k \in \mathbb{N}$ the operator T_k maps $C(A_k, B)$ into $C(A_k, B)$ continuously and from (3.4) and (3.5) for each $f \in U_k(r)$ it follows that the inequality

$$\begin{aligned} \|T_k f\|_{A_k} &\leq P + \sup_{s \in A_k} \int_{A_k} \|W(s, y) f(y)\|_B d\mu_y + \sup_{s \in A_k} \int_{A_k} \|Q(s, y, f(y)) - Q(s, y, 0) + Q(s, y, 0)\|_B d\mu_y \leq \\ &\leq P + \mu(A_k)(W_0(A_x) + L(A_x, r)) \|F_k\|_{A_k} + Q_0 \leq \mu(A_k)(W_0(A_x) + L(A_x, r))r + Q_0 \end{aligned} \quad (3.6)$$

holds, where $W_0(A_x) = W_0(M_x) + W_0(M_x^*)$ and $L(A_x, r) = L(M_x, r) + L(M_x^*, r)$. Since $\lim_{k \rightarrow \infty} \rho(y_{n_k}, y_0) = 0$ then there exists a number $k_0 = k_0(y_0, r)$, such that for every $k \geq k_0$ from (3.6) it follows that the inequalities

$$\mu(A_k)(W_0(A_x) + L(A_x, r))r + Q_0 < r, \quad \mu(A_k)(W_0(A_x) + L(A_x, r)) \leq q < 1 \quad (3.7)$$

hold. Obviously the first of them implies that $T_k(U_k(r)) \subseteq U_k(r)$.

Let $\tilde{k} \geq k_0$ be an arbitrary fixed number and let $f, g \in U_{\tilde{k}}(r)$ be arbitrary functions. If we denote by $F_{\tilde{k}}$ and $G_{\tilde{k}}$ their middle extensions on A_x , then from (3.6) it follows the estimation

$$\begin{aligned} \|T_{\tilde{k}} f - T_{\tilde{k}} g\|_{A_{\tilde{k}}} &\leq \mu(A_{\tilde{k}})(\sup_{s, y \in A_{\tilde{k}}} \|W(s, y)(f(y) - g(y))\|_B + \sup_{s, y \in A_{\tilde{k}}} \|Q(s, y, f(y)) - Q(s, y, g(y))\|_B) \leq \\ &\leq \mu(A_{\tilde{k}})(\sup_{s, y \in A_{\tilde{k}}} \|W(s, y)(F_{\tilde{k}}(y) - G_{\tilde{k}}(y))\|_B + \sup_{s, y \in A_{\tilde{k}}} \|Q(s, y, F_{\tilde{k}}(y)) - Q(s, y, G_{\tilde{k}}(y))\|_B) \leq \\ &\leq \mu(A_{\tilde{k}})(W_0(A_x) + L(A_x, r)) \|F_{\tilde{k}} - G_{\tilde{k}}\|_{A_{\tilde{k}}} = \mu(A_{\tilde{k}})(W_0(A_x) + L(A_x, r)) \|f - g\|_{A_{\tilde{k}}} \leq q \|f - g\|_{A_{\tilde{k}}}. \end{aligned}$$

Therefore the operator $T_{\tilde{k}}$ maps $U_{\tilde{k}}(r)$ into $U_{\tilde{k}}(r)$ and is a contraction. Therefore the equation (2.1) has at least one solution $f \in U_{\tilde{k}}(r)$. $\square$

Theorem 3.6. Let the set $\Omega_* \subset \Omega$ be an arbitrary M -star and M^* -star continuum with $\mu(\Omega_*) < +\infty$ and the following conditions be fulfilled:

1. The conditions (S1) and (S3) hold.
2. Conditions 2 and 3 of Lemma 3.2 hold and for each $x \in \Omega_*$ the sets M_x and M_x^* are connected.

Then for each $p \in C(\Omega_*, B)$ equation (2.1) has exactly one solution $f \in C(\Omega_*, B)$.

Proof: Suppose $f_1, f_2 \in C(\Omega_*, B)$ are two solutions of the equation (2.1). Let $x \in \Omega_*$ be an arbitrary point such that $f_1(x) \neq f_2(x)$. Denote by $r > \sup_{s \in M_x} \|f_1(s)\|_B + \sup_{s \in M_x} \|f_2(s)\|_B > 0$ and by $L(M_x, r) > 0$ the constant, existing according to condition S3. From equation (2.1) and condition S3 it follows that for each $s \in M_x$ the inequality

$$\|f_1(s) - f_2(s)\|_B \leq (W_0(\Omega_*) + L(M_x, r)) \int_{M_x \cup M_x^*} \|f_1(y) - f_2(y)\|_B d\mu_y$$

holds. Using Theorem 2 and Theorem 3 from [7] we get that $\|f_1(s) - f_2(s)\|_B = 0$ for each $s \in M_x$ which contradicts our supposition. $\square$

Everywhere below in our consideration we will assume that $B = \square$ and denote by V^+ the following cone $V^+ = \{f \in C(\Omega, \square) \mid f(x) \geq 0, x \in \Omega\}$.

Theorem 3.7. Let the following conditions be fulfilled:

1. The conditions (S1) and (S4).
2. Conditions 2 and 3 of Lemma 3.2 hold and for each $x \in \Omega$ the sets M_x and M_x^* are connected.
3. For each $x \in \Omega$ there exists a M -star and M^* -star continuum $\Omega_x \subset \Omega$ with $\mu(\Omega_x) < \infty$ such that $M_x, M_x^* \subset \Omega_x$ and $x \in \text{int } \Omega_x$.
4. For each $(x, y) \in \Omega \times M_\Omega$ the operator Q is positive and monotone in the cone V^+ with regard to the order induced by V^+ .
5. For each $(x, y) \in \Omega \times M_\Omega$ the operator W_* have the form $W_* f(x) = \int_{M_x^*} k(x, y) f(y) d\mu_y$, where the function

$k : \Omega \times \Omega \rightarrow \square_+, \square_+ = [0, \infty)$ is continuous in $\Omega \times M_\Omega$ and $f \in C(\Omega, \square)$.

Then for each $p \in V^+$ the equation (4.1) has a unique solution $f \in V^+$ in Ω .

Proof: Let $p \in V^+$ is an arbitrary function, $x \in \Omega$ be an arbitrary fixed point and $\Omega_x \subset \Omega$ is the M -star continuum existing according to condition 3. Denote by $p_x(s) = p(s)$, $s \in \Omega_x$ the restriction of p on Ω_x and by $V_x^+ = \{\varphi \in C(\Omega_x, \square) \mid \varphi(x) \geq 0, x \in \Omega_x\}$ the cone in $C(\Omega_x, \square)$. Obviously condition 4 implies that for each $x \in \Omega$ the operator Q is positive and monotone in the cone V_x^+ with regard to the order induced by V_x^+ . Since $C(\Omega_x, \square)$ is a Banach space and $p_x \in C(\Omega_x, \square)$ then according to Lemma 3.2 the operator T defined by the equality (3.3) maps $C(\Omega_x, \square) \rightarrow C(\Omega_x, \square)$ continuously.

Then, from (2.1), condition 5 and (3.3) for each $s \in \Omega_x$ and $\varphi \in C(\Omega_x, R)$ we have

$$T\varphi(s) = p_x(s) + \int_{M_s} Q(s, y, 0) d\mu_y + \int_{M_s^*} k(s, y) \varphi(y) d\mu_y + \int_{M_s} (Q(s, y, \varphi(y)) - Q(s, y, 0)) d\mu_y \quad (3.8)$$

Let us denote $h_x(s) = p_x(s) + \int_{M_s} Q(s, y, 0) d\mu_y$ and define for every $\varphi \in C(\Omega_x, \square)$ the linear and bounded in

$C(\Omega_x, \square)$ operator $L\varphi(s) = (W_0(\Omega_x) + L(\Omega_x)) \int_{M_s^*} \varphi(y) d\mu_y$, $s \in \Omega_x$, $W_0(\Omega_x) = \sup_{s, y \in \Omega_x} k(s, y)$. Since the operator L

is positive in V_x^+ with regard to the order induced by V_x^+ , then from Theorem 2 of [7] it follows that the spectral radius of the operator L is equal to zero. Therefore the equation $h(s) + L\varphi(s) = \varphi(s)$ has a unique solution $\varphi_x^0 \in V_x^+$. From condition S4 and equality (3.8) it follows that for every $\varphi \in V_x^+$ the inequality $T\varphi(s) \leq h(s) + L\varphi(s)$ holds for each $s \in \Omega_x$. Then we can conclude that the operator T maps the order interval $[0, \varphi_x^0] = \{\varphi \in V_x^+ \mid 0 \leq \varphi(s) \leq \varphi_x^0(s), s \in \Omega_x\}$ into itself and therefore the operator T has at least one fixed point φ_x belonging to the same order interval ([11], Chapter 8). Theorem 3.6 implies that solution φ_x is unique in $C(\Omega_x, \square)$. Define the function f_Ω for every $x \in \Omega$ as follow: $f_\Omega(s) = \varphi_x(s)$, $s \in \Omega_x$. From Theorem 3.6 it follows that for every $x, y \in \Omega$ for which $M_x \cap M_y \neq \emptyset$ for the corresponding unique solutions $\varphi \in C(\Omega_x, \square)$ and $\varphi_y \in C(\Omega_y, \square)$ we have that $\varphi_x(s) = \varphi_y(s)$ for each $s \in \Omega_x \cap \Omega_y$. Therefore $f_\Omega \in C(\Omega, \square)$ and is unique solution of (2.1) in Ω . $\square$

IV. Inequalities

In this section we apply the results obtained in section 3 to study the inequality (2.2).

Definition 4.1. We say that the inequality (2.2) has a local solution in some M -star set $\Omega_* \subset \Omega$ for some $p \in C(\Omega_*, B)$ if there exists a point $x_p \in \Omega_*$ with $\mu(M_{x_p}) > 0$ and a function $g \in C(M_{x_p}, B)$ such that g satisfies the inequality (2.2) for each $s \in M_{x_p}$. If $p, g \in C(\Omega, B)$ and g satisfies the inequality (2.2) for each $x \in \Omega_*$ then we say that g is a solution of (2.2) in $\Omega_* \subset \Omega$.

Theorem 4.2. Let the following conditions be fulfilled:

1. Condition 2 of Theorem 3.6 and condition (S1) hold.
2. The function $Q: \Omega \times M_\Omega \times \square \rightarrow \square$ for every two fixed elements $x \in \Omega$ and $y \in M_\Omega$ is a monotonously increasing function of v .
3. The condition 5 of Theorem 3.7 holds.

Then, if for the functions $f, g \in C(\Omega, \square)$ the inequality

$$g(x) - Tg(x) < f(x) - Tf(x) \quad (4.1)$$

holds for each $x \in \Omega$, then the inequality $g(x) < f(x)$ holds for each $x \in \Omega$ also.

Proof. Denote by Π the following subset of Ω , $\Pi = \{x \in \Omega \mid g(y) < f(y), y \in M_x\}$. Since the maps $M, M^* \in M_0$ and the relation $M^* \prec M$ holds, then condition A4 implies that there exists $x_0 \in \Omega$ such that $\mu(M_{x_0}^*) = 0$ and therefore $\mu(M_{x_0}) = 0$. Then from condition A2 it follows that for each $x \in M_{x_0}$ we have that $\mu(M_x^*) = 0$ and for each $x \in M_{x_0}$, $\mu(M_x) = 0$ too. Since $W_* f(x) = W_* g(x)$ for every $x \in M_{x_0}^*$ and $Kf(x) = Kg(x)$ for each $x \in M_{x_0}$, then (4.1) implies that $x_0 \in \Pi$ and therefore $\Pi \neq \emptyset$.

Let $\{x_n\}_{n=1}^\infty \subset \Pi$ be an arbitrary convergent sequence and denote by $x_0 = \lim_{n \rightarrow \infty} x_n$. If we assume that $x_0 \notin \Pi$, then there exists a point $y \in M_{x_0}$ such that $g(y) \geq f(y)$ and therefore from (4.1) it follows that $Tg(y) - Tf(y) > 0$. Then there exists a set $G \subseteq M_y$, such that $\mu(G) > 0$ and $g(s) > f(s)$ for each $s \in G$. Then for each $n \in \mathbb{N}$ we have that $\mu(G) \leq \mu(M_y \setminus M_{x_n}) \leq \mu(M_{x_0} \setminus M_{x_n})$ and from condition A2 it follows that $\mu(G) = 0$ which contradicts our assumption. Thus we proved that Π is a closed set.

It is simply to see that if $x \in \Pi$ then obviously $M_x \subset \Pi$, i.e. Π is a M -star set. Since $f, g \in C(\Omega, \square)$ then for every $x \in \Pi$ and each $y \in M_x$ there exists an open ball $U(y, \varepsilon_y)$ such that $U(y, \varepsilon_y) \subset \Pi$. Then the set $U_x = \bigcup_{y \in M_x} U(y, \varepsilon_y)$ is an open cover of M_x such that $U_x \subset \Pi$. Because M_x is compact there exist finite

number of balls, such that $\bigcup_{k=1}^n U(y_k, \varepsilon_{y_k}) \subset \Pi$. Then according to Theorem 3.19 [8], there exists $\delta(\varepsilon_{y_1}, \dots, \varepsilon_{y_n}) > 0$ such that for each $s \in \Omega$, $\rho(s, x) < \delta$ we have that $M_s \subset \Pi$ and therefore we proved that Π is an open set.

Since Ω is a connected metric space then we can conclude that $\Pi = \Omega$. Then we have that for every $x \in \Omega$ the inequality $Q(x, y, g(y)) < Q(x, y, f(y))$ holds for each $y \in M_x$. Then from condition 3 and (4.1) it follows that $g(x) < f(x) - Tf(x) + Tg(x) < f(x)$ and therefore $g(x) < f(x)$ for each $x \in \Omega$. $\square$

Remark 4.3. It is simply to see that if we replace the strictly inequality (4.1) by a non-strictly one, the assertion of the theorem still holds for every $x \in \Omega \setminus \text{Ker} M$.

Corollary 4.4. Let the conditions of Theorem 3.7 hold.

Then for each $x \in \Omega$ the inequality $g(x) \leq f(x)$ holds, where $f \in C(\Omega, \square)$ is the unique solution of equation (2.1) and $g \in C(\Omega, \square)$ is an arbitrary solution of the inequality (2.2).

Proof: Let for each $p \in V^+$ is arbitrary. Then according to Theorem 3.7 the equation (2.1) has a unique solution $f \in C(\Omega, \square)$. For every solution $g \in C(\Omega, \square)$ of the inequality (2.2) we have that the inequality

$$g(x) - Tg(x) \leq p(x) = f(x) - Tf(x) \quad (4.2)$$

holds for every $x \in \Omega$. Then if for some $x \in \Omega$ we have that $g(x) - Kg(x) = p(x) = f(x) - Kf(x)$ and therefore $g(x) = f(x)$. For the points $x \in \Omega$ for which the inequality (4.2) is strictly, the assertion of the Corollary 4.4 follows from Theorem 4.2. $\square$

V. Applications to the Integral Inequalities with Maxima

As an illustration of the results obtained in the previous sections we will consider integral equations and corresponding inequalities with maxima. For simplicity and visibility we will consider the two dimensional case ($n = 2$), but all conclusions hold for $n > 3$ too.

Let $\Omega = \square_+^2 = [0, \infty) \times [0, \infty)$, ρ be the Euclidean metric, $B = \square$, μ be the Lebesgue measure, $a, b, c \geq 0$, $ab > 0$ be arbitrary constants and $x_0 = (x_1^0, x_2^0) \in \square_+^2$ be an arbitrary fixed point. Introduce partial order in $\square_+^2$ as follow: for $x, y \in \square_+^2$ we write that $x \leq y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. If $x \leq y$ and $x \neq y$ we write $x < y$.

Consider the following sets: $\Omega(x_0) = \{x \in \square_+^2 \mid x \geq x_0\}$, $J(x_0) = \{[x_1^0 - a, \infty) \times [x_2^0 - b, \infty)\}$ and $J_\Omega(x_0) = J(x_0) \setminus \Omega(x_0)$. Let $M, M^* \in M_0$ for which we have that $\text{Ker} M^* \subseteq \text{Ker} M \subseteq \text{cl } J_\Omega(x_0)$, the sets $H_x \subset J(x_0)$ are compact sets such that for each $x \in \Omega(x_0)$ we have that $x \in H_x$ and $\Omega(x_0) \cap H_x \subseteq M_x$. Consider the equation

$$f(x) = c + \int_{M_x^*} w(x, y) f(y) dy + \int_{M_x} k(x, y) \max_{s \in H_y} f(s) dy \quad (5.1)$$

and the inequality

$$g(x) \leq c + \int_{M_x^*} w(x, y) g(y) dy + \int_{M_x} k(x, y) \max_{s \in H_y} g(s) dy, \quad (5.2)$$

where the functions $w, k \in C(\Omega(x_0) \times \Omega(x_0), \square_+)$ and $g(x) \leq c = f(x)$ for each $J(x_0)$.

It is not hard to verify that all condition of Corollary 4.4 are fulfilled and therefore we have that $g(x) \leq f(x)$ for $x \in J(x_0)$, where $f \in C(J(x_0), \square_+)$ is the unique solution of (5.1) and $g \in C(J(x_0), \square_+)$ is an arbitrary solution of (5.2). The next theorem is a generalization of Theorem 2.3.1 in [6].

Theorem 5.1. Let the following conditions be fulfilled:

1. The functions $w, k \in C(\Omega(x_0), \square_+)$.
2. The functions $\alpha \in C([x_1^0, \infty), \square_+)$ and $\beta \in C([x_2^0, \infty), \square_+)$ are nondecreasing, $\alpha(x_1) \leq x_1$ for $x_1 \in [x_1^0, \infty)$, $\beta(x_2) \leq x_2$ for each $x_2 \in [x_2^0, \infty)$ and $\alpha(x_1^0) = x_1^0$, $\beta(x_2^0) = x_2^0$.
3. The maps M^* and M are defined as follow: $M_x^* = M_x = \emptyset$ for $x \in J(x_0)$ and $M_x^* = [x_1^0, x_1] \times [x_2^0, x_2]$, $M_x = [\alpha(x_1^0), \alpha(x_1)] \times [\beta(x_2^0), \beta(x_2)]$ for every $x \in \Omega(x_0)$.
4. $H_x \subset J(x_0)$ are compact sets such that for each $x \in \Omega(x_0)$ we have that $x \in H_x$ and $\Omega(x_0) \cap H_x \subseteq [\alpha(x_1^0), \alpha(x_1)] \times [\beta(x_2^0), \beta(x_2)]$.
5. The function $g \in C(J(x_0), \square_+)$ for every $x \in \Omega(x_0)$ satisfies the inequality

$$g(x) \leq c + \int_{M_x^*} w(y) g(y) dy + \int_{M_x} k(y) \max_{s \in H_y} g(s) dy \quad (5.3)$$

and the inequality $g(x) \leq c$ for $x \in J_\Omega(x_0)$.

Then for $x \in J(x_0)$ the inequality

$$g(x) \leq c \exp\left(\int_{M_x^*} w(y) dy + \int_{M_x} k(y) dy\right) \quad (5.4)$$

holds.

Proof: Conditions 2 and 3 implies that $\text{Ker} M^* \subseteq \text{Ker} M$. Then it is not difficult to see that all condition of Corollary 4.4 are fulfilled and therefore the inequality $g(x) \leq f(x)$ holds, for $x \in J(x_0)$ where $f \in C(J(x_0), \square_+)$ is the unique solution of the equation

$$f(x) = c + \int_{M_x^*} w(y) f(y) dy + \int_{M_x} k(y) \max_{s \in H_y} f(s) dy \quad (5.5)$$

From (5.5) and condition 4 for $x \in \Omega(x_0)$ it follows that the inequality

$$f(x) \leq c + \int_{M_x^*} w(y) \max_{s \in H_y} f(s) dy + \int_{M_x} k(y) \max_{s \in H_y} f(s) dy \quad (5.6)$$

holds. Let denote $\varphi(x) = \max_{y \in H_x} f(y)$. Condition 4 implies that for each $x \in \Omega(x_0)$ we have that $f(x) \leq \varphi(x)$. Let $x \in \Omega(x_0)$ be an arbitrary fixed point. Then if $\max_{y \in M_x^*} \max_{s \in H_y} f(s) = f(x)$ then from (5.6) it follows that the inequality

$$\varphi(x) \leq c + \int_{M_x^*} w(y) \varphi(y) dy + \int_{M_x} k(y) \varphi(y) dy \quad (5.7)$$

holds. If $\max_{y \in M_x^*} \max_{s \in H_y} f(s) = f(y_0)$, $y_0 \in M_x^*$, $y_0 \neq x$ then

$$\varphi(x) = \max_{y \in H_x} f(y) \leq f(y_0) = c + \int_{M_{y_0}^*} w(y)\varphi(y)dy + \int_{M_{y_0}} k(y)\varphi(y)dy \leq c + \int_{M_x^*} w(y)\varphi(y)dy + \int_{M_x} k(y)\varphi(y)dy$$

and therefore the inequality (5.7) holds for each $x \in \Omega(x_0)$.

Let $x \in \Omega(x_0)$ be an arbitrary fixed point and let $z \in \Omega(x_0)$ be an arbitrary point for which we have $x \leq z$. Then we have that $M_x^*, M_x \subseteq B_z = [0, z_1] \times [0, z_2]$ for each $s \in B_z$. Denote by $K_M = \max_{s \in M_x} k(s)$ and by $C_z = \max_{y \in B_z} k(y)$

and let $\varepsilon > 0$ be an arbitrary number. Condition A2 implies that for the point $x \in B_z$ there exists $s_x \in B_z$ with $s_x \geq x$ such that $\mu(M_{s_x} \setminus M_x)C_z < \varepsilon$. Let $h \in C(B_z, [0, 1])$ is an Urisohn function for the compact sets M_x and $B_z \setminus \text{int } M_{s_x}$. Then for the continuous function $k_x(s) = h(s)k(s)$ we have that

$$k_x(s) = \begin{cases} k(s), & s \in M_x \\ 0, & s \in B_z \setminus \text{int } M_{s_x} \end{cases}$$

Then from (5.7) it follows that for each $s \in B_z$ the inequality

$$\varphi(s) \leq c + \int_{M_s^*} q_x(y)\varphi(y)dy \quad (5.8)$$

holds, where $q_x(s) = w(s) + k_x(s)$.

Consider the equation

$$\phi(s) = 1 + \int_{M_s^*} q_x(y)\phi(y)dy \quad (5.9)$$

Then from the results in [7] it follows that for each solution $\varphi \in C(B_z, \mathbb{R}_+)$ of the inequality (5.8) the inequality $\varphi(s) \leq c\phi(s)$, $s \in B_z$ holds, where $\phi(s)$ is the unique continuous solution in B_z of the equation (5.9) when $c = 1$. Moreover for each $s \in B_z$ the following estimation

$$\phi(s) \leq \exp\left(\int_{M_s^*} q_x(y)dy\right) \quad (5.10)$$

holds. Therefore from (5.10) it follows that for each solution $\varphi \in C(B_z, \mathbb{R}_+)$ of the inequality (5.8) we have that the estimation

$$\varphi(s) \leq c\phi(s) \leq c \exp\left(\int_{M_s^*} q_x(y)dy\right) \quad (5.11)$$

is fulfilled for every $s \in B_z$. Particularly for $s = x$ from (5.11) it follows that

$$\varphi(x) \leq c \exp\left(\int_{M_x^*} w(y)dy + \int_{M_x} k(y)dy + \int_{M_{s_x} \setminus M_x} k_x(y)dy\right) \leq c \exp\left(\int_{M_x^*} w(y)dy + \int_{M_x} k(y)dy + \varepsilon\right) \quad (5.12)$$

Since in the inequality (5.12) $\varepsilon > 0$ is arbitrary then passing $\varepsilon \rightarrow 0^+$ we receive that the inequality $g(x) \leq c \exp\left(\int_{M_x^*} w(y)dy + \int_{M_x} k(y)dy\right)$ holds. $\square$

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