



A Result on Generalized Harmonic Alternating Series

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Abstract: In this paper, the generalization of an alternating harmonic series and its convergent in a particular case (for even) is discussed.

Key Words: Alternating series, Alternating Harmonic Series, Uniformly Convergent.

I. Introduction

A sequence is a list of numbers that are written in a particular order. If the sequence goes on forever then it is called an infinite sequence denoted by $\{a_n\}_{n=1}^{\infty}$ otherwise it is called a finite sequence. A sequence usually has a rule, which is a way to find the value of each term. So, we want a formula with "n" in it (where n is any term number). A sequence is a function whose domain is the set of natural numbers and co-domain is the set of real numbers. This is an important use in sequences and series. By assuming the sequence terms as function evaluation will allow us to do many things with the sequence and otherwise it could not possible. There are some special sequences like Arithmetic sequences, Geometric sequences, Harmonic sequences, Triangular Number sequence, Fibonacci sequence, Alternating sequence, Alternating Harmonic sequences, etc. The sum of infinite terms that follow a rule is called a series. There are so many types of series based the different types of sequences. In this paper, a result on the Sum of an Alternating Harmonic series is proved. We mention here some definitions, formula and results which have been used in the theorem

II. Preliminary Results

Definition1.1: If for the given sequence $\{a_n\}_{n=1}^{\infty}$, a function $f(x)$ such that $f(x) = a_n$ and $\lim_{n \rightarrow \infty} f(x) = L$ then $\lim_{n \rightarrow \infty} a_n = L$.

Definition1.2: The sequence $\{a_n\}_{n=1}^{\infty}$ is called increasing if $a_n \leq a_{n+1}$ for every n.

Definition1.3: The sequence $\{a_n\}_{n=1}^{\infty}$ is called decreasing if $a_n \geq a_{n+1}$ for every n.

Definition1.4: The sequence $\{a_n\}_{n=1}^{\infty}$ is an increasing or decreasing sequence, then it is called a monotonic sequence.

Definition1.5: For the sequence $\{a_n\}_{n=1}^{\infty}$, if there exist a number m such that $m \leq a_n$ for every n, we say that the sequence is called bounded below and m is called a lower bound for the sequence.

Definition1.6: For the sequence $\{a_n\}_{n=1}^{\infty}$, if there exist a number M such that $a_n \leq M$ for every n, we say that the sequence is called bounded above and M is called an upper bound for the sequence.

Definition1.7: If any sequence is both bounded above and bounded below then it is called a bounded sequence.

Theorem1.8: If the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and monotonic then $\{a_n\}_{n=1}^{\infty}$ is convergent.

Definition1.9: If $\{a_n\}_{n=1}^{\infty}$ is any sequence then the sum $\sum_{i=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$ called an infinite series.

Definition1.10: If $s_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^n a_i$ then s_n are called partial sum and $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = \sum_{i=1}^{\infty} a_n$.

Definition1.11: A sequence of functions $\{f_n\}$, $n = 1, 2, 3, \dots$ is said to be uniformly convergent to f for a set E of values of x if, for each $\epsilon > 0$, an integer N can be found such that $|f_n(x) - f(x)| < \epsilon$, for $n \geq N$ and all $x \in E$.

A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{k=1}^n f_k(x) = S_n(x)$$

converges uniformly on E .

Definition1.12:

A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \dots$$

Or,

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + \dots$$

Where a_n are positive, is called an alternating series.

Definition1.13: A series $\sum a_n$ converges absolutely if the series $\sum |a_n|$ converges.

Definition1.14: A series is conditionally convergent if it converges but does not converge absolutely.

Result1.15: We have for $m, n \in \mathbb{N}$ and $m < 2n$,

$$\int \frac{x^{m-1}}{1-x^{2n}} dx = \frac{1}{2n} \{(-1)^{m+1} \ln(1+x) - \ln(1-x)\} - \frac{1}{2n} \sum_{k=1}^{n-1} \cos\left(\frac{km}{n}\pi\right) \ln\left(1 - 2x \cos\left(\frac{k}{n}\pi\right) + x^2\right) + \frac{2}{n} \sum_{k=1}^{n-1} \sin\left(\frac{km}{n}\pi\right) \arctan\left(\frac{x - \cos\frac{k}{n}\pi}{\sin\frac{k}{n}\pi}\right)$$

Theorem: For any even integer $a > 1$ and an integer $b, 0 < b < a, K(a, b) = \sum_{k=1}^{\infty} \frac{b}{ak(ak-b)}$ sums to

$$\frac{1}{a} \{(-1)^{b+1} \ln 2 + \ln a - \sum_{k=1}^{\frac{a-2}{2}} \{2 \ln(2 \sin \pi \frac{k}{a}) \cos(\frac{2kb\pi}{a}) + (\frac{a-2k}{2a} \pi) \sin(\frac{2kb}{a} \pi)\}\}$$

Proof:

Let $0 < x < t < 1$,

$$\text{Consider } \sum_{k=1}^{\infty} x^{ak-b-1} = \frac{x^{a-b-1}}{1-x^a} \quad (1)$$

The series converges uniformly on $[0, t]$. Integrate term by term over $[0, t]$ and

Let $a=2n, m=a-b$ and $x=t$ in Result 1.15

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{t^{ak-b}}{ak-b} &= \frac{1}{a} \{(-1)^{a-b+1} \ln(1+t) - \ln(1-t)\} - \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{k(a-b)2\pi}{a}\right) \ln\left(1 - 2t \cos\left(\frac{2k\pi}{a}\right) + t^2\right) \\ &\quad + \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{k(a-b)2\pi}{a}\right) \arctan\left(\frac{t - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{k(a-b)2\pi}{a}\right) \arctan\left(-\cot\frac{2\pi k}{a}\right) \\ &= \frac{1}{a} \{(-1)^{b+1} \ln(1+t) - \ln(1-t)\} - \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{kb2\pi}{a}\right) \ln\left(1 - 2t \cos\left(\frac{2k\pi}{a}\right) + t^2\right) \\ &\quad + \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{bk2\pi}{a}\right) \left\{ \arctan\left(\frac{t - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \arctan\left(-\cot\frac{2\pi k}{a}\right) \right\} \end{aligned} \quad (2)$$

Now consider

$$\sum_{k=1}^{\infty} x^{ak-1} = \frac{x^{a-1}}{1-x^a}$$

The series convergent uniformly over $[0, t]$. Integrating them term by term over $[0, t]$, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{t^{ak}}{ak} &= \int_0^t \frac{x^{a-1}}{1-x^a} dx \\ \Rightarrow \sum_{k=1}^{\infty} \frac{t^{ak}}{ak} &= -\frac{1}{a} \ln(1-t^a) \end{aligned} \quad (3)$$

Now consider

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{t^{ak-b}}{ak-b} - \frac{t^{ak}}{ak} \right) &= \frac{1}{a} \left\{ (-1)^{b+1} \ln(1+t) + \ln \frac{1-t^a}{1-t} \right\} - \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{kb2\pi}{a}\right) \ln\left(1 - 2t \cos\left(\frac{2k\pi}{a}\right) + t^2\right) \\ &\quad + \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{bk2\pi}{a}\right) \left\{ \arctan\left(\frac{t - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \arctan\left(-\cot\frac{2\pi k}{a}\right) \right\} \\ &= \frac{1}{a} \{(-1)^{b+1} \ln(1+t) + \ln(1+t+t^2+t^3+\dots+t^{a-1})\} \\ &\quad - \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{kb2\pi}{a}\right) \ln\left(1 - 2t \cos\left(\frac{2k\pi}{a}\right) + t^2\right) + \frac{1}{a} \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{bk2\pi}{a}\right) \left\{ \arctan\left(\frac{t - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \arctan\left(-\cot\frac{2\pi k}{a}\right) \right\} \end{aligned}$$

Letting $t \rightarrow 1$ in the above

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{1}{ak-b} - \frac{1}{ak} \right) &= \frac{1}{a} \{(-1)^{b+1} \ln 2 + \ln a - \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{kb2\pi}{a}\right) \ln(2 - 2\cos\left(\frac{2k\pi}{a}\right)) \\ &\quad + \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{bk2\pi}{a}\right) \left\{ \arctan\left(\frac{1 - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \arctan\left(-\cot\frac{2\pi k}{a}\right) \right\}\} \\ \Rightarrow \sum_{k=1}^{\infty} \frac{b}{(ak-b)ak} &= \frac{1}{a} \{(-1)^{b+1} \ln 2 + \ln a - \sum_{k=1}^{\frac{a-2}{2}} \cos\left(\frac{kb2\pi}{a}\right) \ln(2 - 2\cos\left(\frac{2k\pi}{a}\right)) \\ &\quad + \sum_{k=1}^{\frac{a-2}{2}} \sin\left(\frac{bk2\pi}{a}\right) \left\{ \arctan\left(\frac{1 - \cos\frac{2\pi k}{a}}{\sin\frac{2\pi k}{a}}\right) - \arctan\left(-\cot\frac{2\pi k}{a}\right) \right\}\} \end{aligned}$$

$$= \frac{1}{a} \{ (-1)^{b+1} \ln 2 + \ln a - \sum_{k=1}^{\frac{a-2}{2}} 2 \cos \left(\frac{kb2\pi}{a} \right) \ln \left(2 \sin \frac{k\pi}{a} \right) + \sum_{k=1}^{\frac{a-2}{2}} \sin \left(\frac{bk2\pi}{a} \right) \left\{ \frac{\pi}{2} - \frac{\pi k}{a} \right\} \} \quad \therefore$$

$$\sum_{k=1}^{\infty} \frac{b}{(ak-b)ak} = \frac{1}{a} \{ (-1)^{b+1} \ln 2 + \ln a - \sum_{k=1}^{\frac{a-2}{2}} [2 \cos \left(\frac{kb2\pi}{a} \right) \ln \left(2 \sin \frac{k\pi}{a} \right) + \left(\frac{a-2k}{2a} \right) \pi \sin \left(\frac{bk2\pi}{a} \right)] \}.$$

Example: Let $a=4$ and $b=3$, then the series

$$\therefore \sum_{k=1}^{\infty} \frac{3}{(4k-3)4k} = \frac{1}{4} \left[3 \ln 2 + \frac{\pi}{4} \right].$$

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