



On Holomorphically Subprojective Kaehlerian Manifold of First Kind

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Abstract: Two types of holomorphically subprojective Kaehlerian manifolds, namely of first and second kind come into existence through a series of papers by S. Yamaguchi and others when they were trying for complexification of concept of subprojective Riemannian manifold. The properties of one of these two types of manifolds are studied in the present paper and a number of interesting and meaningful results are obtained.

Keywords: Holomorphically Subprojective, Kaehlerian Manifold, Structure tensor, Metric tensor

I. INTRODUCTION

In a real $2n$ -dimensional Kaehlerian manifold M admitting the structure tensor F and the metric tensor g satisfying [7].

$$F_i^h F_h^j = -\delta_i^j, \quad g_{ji} F_p^q F_q^i = g_{pq} \text{ and } \nabla_k F_j^h,$$

the curve $x^h = x^h(t)$ satisfying the differential equation

$$\frac{d^2 x^h}{dt^2} + \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^i}{dt} = \alpha \frac{dx^h}{dt} + \beta F_r^h \frac{dx^r}{dt} \gamma v^h + \epsilon \tilde{v}^h$$

where the indices h, i, j, k run over the range $1, 2, \dots, 2n$, $\alpha, \beta, \gamma, \epsilon$ are functions of t , v is a vector field and $\tilde{v}^h = F_r^h v^r$ is complex transform of v , is called holomorphically subplane curve [4]. Further the Kaehlerian manifold $M(2n \geq 6)$ admitting the real coordinate system (x^h) is said to be holomorphically subprojective Kaehlerian manifold [5] iff its Christoffel symbol takes the form

$$\left\{ \begin{matrix} h \\ ji \end{matrix} \right\} = \rho_j \delta_i^h + \rho_i \delta_j^h + \tilde{\rho}_j F_i^h + \tilde{\rho}_i F_j^h + f_{ji} \zeta^h - f_{jr} F_i^r \zeta^h$$

where ρ_i and f_{ji} are respectively the components of a covector and symmetric tensor satisfying

$$f_{ij} F_i^r = f_{ri} F_j^r$$

and ζ^h are the components of contravariant analytic vector field.

II. HOLOMORPHICALLY SUBPROJECTIVE KAEHLERIAN MANIFOLD

A straightforward, rather lengthy & complicated calculation based upon the nature and form of Christoffel symbol gives [5]

$$(2.1) \quad R_{kjih} = \left(\bar{a} + \frac{\lambda + \epsilon}{2(n-1)} |\zeta|^2 \right) (g_{ik} g_{jh} - g_{ij} g_{kh} - F_{ik} F_{jh} + F_{ij} F_{kh} + 2F_{ih} F_{kj}) \\ - \frac{(\lambda + \epsilon)}{2(n-1)} \left\{ (F_{ik} \bar{\zeta}_j - F_{ij} \bar{\zeta}_k + 2F_{jk} \bar{\zeta}_i - g_{ij} \zeta_k + g_{ik} \zeta_j) \bar{\zeta}_h \right. \\ - (F_{ik} \zeta_j - F_{ij} \zeta_k + 2F_{jk} \bar{\zeta}_i + g_{ij} \bar{\zeta}_k + g_{ik} \bar{\zeta}_j) \bar{\zeta}_h \\ + \zeta_i (\zeta_k g_{jh} - \zeta_j g_{kh} - \bar{\zeta}_k F_{jh} + \bar{\zeta}_j F_{kh}) \\ + \bar{\zeta}_i (\bar{\zeta}_k g_{jh} - \bar{\zeta}_j g_{kh} + \zeta_k F_{jh} - \zeta_j F_{kh}) \\ \left. - 2(F_{ih} \bar{\zeta}_k \zeta_j + 2F_{ih} \bar{\zeta}_j \zeta_k) \right\} \\ - \frac{2\lambda + (n+1)\epsilon}{(n-1)|\zeta|^2} (\zeta_k \bar{\zeta}_j \bar{\zeta}_i \zeta_h - \bar{\zeta}_k \zeta_j \bar{\zeta}_i \zeta_h - \zeta_k \bar{\zeta}_j \bar{\zeta}_h \zeta_i + \bar{\zeta}_k \zeta_j \bar{\zeta}_h \zeta_i)$$

$$(2.2) \quad R_{kh} = - \left[2(n+1)\bar{a} + \frac{n(\lambda + \epsilon)}{(n-1)} |\zeta|^2 \right] g_{kh} + \frac{n\lambda + \epsilon}{n-1} (\zeta_k \zeta_h + \bar{\zeta}_k \bar{\zeta}_h), \text{ and}$$

$$(2.3) \quad R = -4n(n+1)\bar{a} - 2(n\lambda + (n+1)\epsilon) |\zeta|^2$$

as expressions for curvature tensor, Ricci tensor and scalar curvature respectively, where we have used

$$\lambda \zeta_k = U_{kr}^r, \quad \epsilon |\zeta|^2 \bar{\zeta}_i = U_{kri} \bar{\zeta}^k \zeta^r, \quad U_{rk} \bar{\zeta}^r = \bar{a} \zeta_k$$

$$U_{kji} = F_{ij} \alpha_k + F_{ij} F_k^r F_{ik} \alpha_j - F_{ij} F_j^r + F_i^r F_{rj} \alpha_k - F_i^r F_{kr} \alpha_j + \\ + F_i^r F_{rj} F_k^r - F_j^r F_{kr} F_i^r + \nabla_k F_{ji} - \nabla_j F_{ki}$$

$$U_{kj} = \nabla_k \rho_j + \rho_k \rho_j - \bar{\rho}_k \bar{\rho}_j + (\rho' - a) F_{kj} - (\rho'' + b) F_{kr} F_j^r$$

$$F_k' = F_{kr}\xi^r, \quad F_k'' = F_{kr}\bar{\xi}^r, \quad \rho' = \rho_r\xi^r \text{ and } \rho'' = \rho_r\bar{\xi}^r$$

From (2.2), it is evident that the Ricci tensor of holomorphically sub projective Kaehlerian manifold has taken the form

$$R_{kh} = Ag_{kh} + B(\xi_k\xi_h + \bar{\xi}_k\bar{\xi}_h)$$

and so we may state-

Theorem (2.1):- Every holomorphically subprojective Kaehlerian manifold is quasi-Einstein manifold [1].

Since the vector field ξ is a K-torse forming vector field [6], we immediately have

$$(2.4) \quad (a) \quad \nabla_j \xi_k = ag_{ki} + bF_{jk} + \alpha_j \xi_k + \bar{\alpha}_j \bar{\xi}_k$$

$$(b) \quad \nabla_j \bar{\xi}_k = aF_{jk} + bg_{jk} + \alpha_j \bar{\xi}_k + \bar{\alpha}_j \xi_k$$

and so when (2.2) is differentiated covariantly and values are substituted from (2.4), then we find

$$(2.5) \quad \nabla_l R_{kh} = \frac{(n\lambda + \epsilon)}{(n-1)} [2\alpha_l(\xi_k\xi_h + \bar{\xi}_k\bar{\xi}_h) + a(g_{lh}\xi_k + g_{lk}\xi_h + F_{lh}\bar{\xi}_k + F_{lk}\bar{\xi}_h) + b(F_{lh}\xi_k + F_{lk}\xi_h - g_{lh}\bar{\xi}_k - g_{lk}\bar{\xi}_h)]$$

which under the assumption that the manifold is Ricci-symmetric and multiplication by $\xi^h\xi^k$ gives

$$\alpha_l + a\xi_l + b\bar{\xi}_l = 0$$

showing that

Theorem (2.2):- If holomorphically subprojective Kaehlerian manifold is Ricci symmetric then the associated vector α of ξ is a linear combination of ξ and $\bar{\xi}$.

Further from (2.2) we may obtain

$$R_{kh}\xi^k\xi^h = -[2(n+1)\bar{a} + \epsilon|\xi|^2]|\xi|^2$$

$$R_{kh}\bar{\xi}^k\bar{\xi}^h = -[2(n+1)\bar{a} + \epsilon|\xi|^2]|\xi|^2$$

and consequently we may state.

Theorem (2.3):- In a holomorphically subprojective Kaehlerian manifold the Ricci tensor has α as eigen values and corresponding eigen vector are ξ and $\bar{\xi}$.

Theorem (2.4):- The Ricci curvatures of a holomorphically subprojective Kaehlerian manifold with respect to the directions ξ and $\bar{\xi}^h$ are identical.

III. HOLOMORPHICALLY SUBPROJECTIVE KAEHLERIAN MANIFOLD OF FIRST KIND

Holomorphically subprojective Kaehlerian manifold of first kind are characterized by the condition

$$2\lambda + (n+1) \in 0$$

and consequently for such manifolds the expressions for curvature tensor, Ricci tensor and scalar curvature take the form

$$(3.1) \quad R_{kjih} = \left(a - \frac{\epsilon}{4}|\xi|^2 \right) (g_{jh}g_{ki} - g_{kh}g_{ji} + F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih})$$

$$+ \frac{\epsilon}{4} [F_{ik}(\bar{\xi}_j\xi_h - \xi_j\bar{\xi}_h) - F_{ij}(\bar{\xi}_k\xi_h - \xi_k\bar{\xi}_h) + 2F_{jk}(\bar{\xi}_i\xi_h - \xi_i\bar{\xi}_h)$$

$$+ F_{jh}(\xi_k\bar{\xi}_i - \bar{\xi}_k\xi_i) - F_{kh}(\bar{\xi}_i\xi_j - \xi_i\bar{\xi}_j) + 2F_{ih}(\xi_k\bar{\xi}_j - \bar{\xi}_k\xi_j)$$

$$+ g_{jh}(\xi_i\xi_k + \bar{\xi}_i\bar{\xi}_k) - g_{kh}(\xi_i\xi_j + \bar{\xi}_i\bar{\xi}_j) - g_{ij}(\xi_k\xi_h + \bar{\xi}_k\bar{\xi}_h)$$

$$+ g_{ik}(\xi_j\xi_h + \bar{\xi}_j\bar{\xi}_h)]$$

$$(3.2) \quad R_{kh} = -2\bar{a}(n+1)g_{kh} + \frac{n\epsilon}{2}g_{kh}|\xi|^2 - \left(\frac{n+2}{2} \right) (\xi_k\xi_h + \bar{\xi}_k\bar{\xi}_h) \in$$

$$(3.3) \quad R = -4n(n+1)\bar{a} + (n-2)(n+1) \in |\xi|^2$$

and multiplication of (3.2) by F_m^k gives

$$(3.4) \quad S_{ki} = -2\bar{a}(n+1)F_{ki} + \frac{n\epsilon}{2}F_{ki}|\xi|^2 - \left(\frac{n+2}{2} \right) (\xi_k\bar{\xi}_i + \bar{\xi}_k\xi_i)$$

From (3.2), it is clear that Ricci tensor is of the form

$$R_{kh} = ag_{kh} + b(\xi_k\xi_h + \bar{\xi}_k\bar{\xi}_h)$$

where $b \neq 0$, since $\epsilon \neq 0$ and therefore, we may state

Theorem (3.1):- Holomorphically subprojective Kaehlerian manifold of first kind is quasi-Einstein manifold.

Now, the well known H-concircular curvature tensor C, H-Projective curvature tensor P, H-conharmonic curvature tensor D [3] in such spaces will be given by

$$(3.5) \quad C_{kjih} = -\frac{\epsilon}{2n}|\xi|^2(g_{jh}g_{ki} - g_{kh}g_{ji} + F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih})$$

$$+ \frac{\epsilon}{4} [(F_{ik}\bar{\xi}_j - F_{ij}\bar{\xi}_k + 2F_{jk}\bar{\xi}_i - g_{ij}\xi_k + g_{ki}\xi_j)\xi_h$$

$$- (F_{ik}\xi_j - F_{ij}\xi_k + 2F_{jk}\xi_i + g_{ji}\bar{\xi}_k - g_{ik}\bar{\xi}_j)\bar{\xi}_h]$$

$$\begin{aligned}
 & +g_{jh}(\xi_i \xi_k + \bar{\xi}_i \bar{\xi}_k) - g_{kh}(\xi_i \xi_j + \bar{\xi}_i \bar{\xi}_j) + F_{jh}(\xi_k \bar{\xi}_i + \bar{\xi}_k \xi_i) \\
 & - F_{kh}(\xi_i \bar{\xi}_j + \bar{\xi}_i \xi_j) + 2F_{ih}(\xi_k \bar{\xi}_j - \bar{\xi}_k \xi_j)] \\
 (3.6) \quad P_{kjih} = & -\frac{1}{n+1} \frac{\epsilon}{4} [|\xi|^2 (g_{jh}g_{ki} - g_{kh}g_{ji} + F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih} \\
 & + g_{jh}(\xi_k \bar{\xi}_i + \bar{\xi}_k \bar{\xi}_i) - g_{kh}(\xi_i \bar{\xi}_j + \bar{\xi}_i \xi_j) + F_{jh}(\xi_k \bar{\xi}_i - \bar{\xi}_k \xi_i) \\
 & - F_{kh}(\xi_i \bar{\xi}_j + \bar{\xi}_i \xi_j) + 2F_{ih}(\xi_k \bar{\xi}_j - \bar{\xi}_k \xi_j)] \\
 & + \frac{\epsilon}{4} [F_{ik}(\xi_h \bar{\xi}_j - \bar{\xi}_j \bar{\xi}_h) - F_{ij}(\xi_k \bar{\xi}_h - \bar{\xi}_k \xi_h) + 2F_{jk}(\xi_h \bar{\xi}_i - \bar{\xi}_h \xi_i) \\
 & - g_{ij}(\xi_k \bar{\xi}_h + \bar{\xi}_k \bar{\xi}_h) + g_{ik}(\xi_j \bar{\xi}_h + \bar{\xi}_j \bar{\xi}_h)]
 \end{aligned}$$

and

$$(3.7) \quad D_{kjih} = -\frac{1}{(n+2)} \left[n\bar{a} - \frac{(n-2)}{4} \epsilon |\xi|^2 \right] (g_{jh}g_{ki} - g_{kh}g_{ji} + F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih})$$

respectively, where as it is well known that holomorphically subprojective Kaehlerian manifold has zero Bochner curvature[5].

From (3.7), it is quite clear that $\nabla_l D_{kjih} = 0$ showing that the manifold is H-conharmonic symmetric and therefore

Theorem (3.2):- Holomorphically subprojective Kaehlerian manifold of first kind is always H-conharmonic symmetric.

Now, since Holomorphically subprojective Kaehlerian manifold is of zero Bochner curvature and it is H-conharmonic symmetric too, we can find

$$\nabla_l C_{kjih} = \nabla_l R_{kjih}$$

in view of the result

$$D_{kjih} = B_{kjih} + \frac{n}{n+2} (C_{kjih} - R_{kjih})$$

and consequently, we have

Theorem (3.3):- The holomorphically subprojective Kaehlerian manifold of first kind is H-concircularly symmetric (recurrent) iff it is symmetric (recurrent).

Now, if the manifold be supposed to be H-projectively flat [1], the R.H.S. of (3.6) must vanish identically, which on transvection with g^{ji} and g^{kh} is turn gives $|\xi| = 0$ which is impossible and so, we have

Theorem (3.4):- The holomorphically subprojective Kaehlerian manifold of first kind can never be HP-flat.

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