



Solution of Linear Second Order Forward Difference Equation with Variable Coefficients

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Abstract: In this paper, Linear Second Order Forward Difference Equation with Variable Coefficients be solved in special case.

Keywords: Integrating factor, Known integral, step size, forward difference, factorial function.

I. Introduction

Solution of linear second order forward difference equation with variable coefficients of the form

$$\Delta^2 y(x) + P(x)\Delta y(x) + Q(x)y(x) = R(x),$$

where $P(x), Q(x), R(x)$ are functions of x containing step size h also, such that $h \neq 0$ and Δ is forward difference operator, is given by $y(x) = u(x)v(x)$

Where $u(x)$ is given by $\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0$

and $v(x)$ is given by

$$\Delta v(x) = \frac{1}{z(x)} \Delta^{-1} \left\{ z(x+h) \frac{R(x)}{u(x+2h)} \right\} + \frac{c}{z(x)}, \quad c \text{ is an arbitrary constant}$$

$$\text{Where } z(x) = e^{\Delta^{-1} \log \left\{ \frac{1}{1-I(x)} \right\}}$$

$$\text{and } I(x) = \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)}.$$

Method and Proof. Consider a linear second order forward difference equation with variable coefficients of the form

$$\Delta^2 y(x) + P(x)\Delta y(x) + Q(x)y(x) = R(x) \quad \dots(1)$$

where $P(x), Q(x), R(x)$ are functions of x containing step size h also, such that $h \neq 0$ and Δ is forward difference operator.

Let $y(x) = u(x)v(x)$

$$\Delta y(x) = u(x+h)\Delta v(x) + v(x)\Delta u(x)$$

$$\Delta^2 y(x) = \Delta \{ \Delta y(x) \}$$

$$= \Delta \{ u(x+h)\Delta v(x) \} + \Delta \{ \Delta u(x) \cdot v(x) \}$$

$$= u(x+2h)\Delta^2 v(x) + \Delta v(x)\Delta u(x+h) + \Delta u(x+h)\Delta v(x) + v(x)\Delta^2 u(x)$$

$$= u(x+2h)\Delta^2 v(x) + 2\Delta u(x+h)\Delta v(x) + v(x)\Delta^2 u(x)$$

Putting the values of $y(x), \Delta y(x), \Delta^2 y(x)$ in (1), then

$$u(x+2h)\Delta^2 v(x) + 2\Delta u(x+h)\Delta v(x) + v(x)\Delta^2 u(x)$$

$$+ P(x) \{ u(x+h)\Delta v(x) + v(x)\Delta u(x) \} + Q(x)u(x)v(x) = R(x)$$

$$u(x+2h)\Delta^2 v(x) + \{2\Delta u(x+h) + P(x)u(x+h)\}\Delta v(x) \\ + \{\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x)\}v(x) = R(x)$$

$$\text{Take } \Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0 \quad \dots(2)$$

$$\text{Then } u(x+2h)\Delta^2 v(x) + \{2\Delta u(x+h) + P(x)u(x+h)\}\Delta v(x) = R(x)$$

$$\Delta^2 v(x) + \left\{ \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)} \right\} \Delta v(x) = \frac{R(x)}{u(x+2h)}$$

$$\Delta^2 v(x) + I(x)\Delta v(x) = \frac{R(x)}{u(x+2h)} \quad \dots(3)$$

$$\text{Where } I(x) = \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)}$$

$$\text{Take } I(x) = \frac{\Delta z(x)}{z(x+h)} \quad \dots(4)$$

Then,

$$\text{From (3)} \quad \Delta^2 v(x) + \frac{\Delta z(x)}{z(x+h)} \Delta v(x) = \frac{R(x)}{u(x+2h)}$$

$$z(x+h)\Delta^2 v(x) + \Delta v(x)\Delta z(x) = z(x+h) \frac{R(x)}{u(x+2h)}$$

$$\Delta \{z(x)\Delta v(x)\} = z(x+h) \frac{R(x)}{u(x+2h)}$$

$$\Delta v(x) = \frac{1}{z(x)} \Delta^{-1} \left\{ z(x+h) \frac{R(x)}{u(x+2h)} \right\} + \frac{c}{z(x)} \quad \dots(5)$$

Where c is an arbitrary constant and $z(x)$ is an integrating factor of (3)

$$\text{From (4)} \quad I(x) = \frac{\Delta z(x)}{z(x+h)}$$

$$\frac{\Delta z(x)}{z(x+h)} = I(x)$$

$$\frac{z(x+h) - z(x)}{z(x+h)} = I(x)$$

$$1 - \frac{z(x)}{z(x+h)} = I(x)$$

$$\frac{z(x)}{z(x+h)} = 1 - I(x)$$

$$\frac{z(x+h)}{z(x)} = \frac{1}{1 - I(x)}$$

$$\log z(x+h) - \log z(x) = \log \left\{ \frac{1}{1 - I(x)} \right\}$$

$$\Delta \log z(x) = \log \left\{ \frac{1}{1 - I(x)} \right\}$$

$$\log z(x) = \Delta^{-1} \log \left\{ \frac{1}{1 - I(x)} \right\} + k, \text{ where } k \text{ is an arbitrary constant}$$

Taking $k = 0$, because integrating factor has no arbitrary constant

Then,

$$\log z(x) = \Delta^{-1} \log \left\{ \frac{1}{1-I(x)} \right\}$$

$$z(x) = e^{\Delta^{-1} \log \left\{ \frac{1}{1-I(x)} \right\}}$$

$$\text{Where } I(x) = \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)}$$

Example 1. Prove that $u(x) = x^{(n)}$ is a solution of equation

$$\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0,$$

$$\text{if } n(n-1)h^2 + nh(x-(n-2)h)P(x) + (x-(n-2)h)(x-(n-1)h)Q(x) = 0$$

where $n \in N$, N is the set of all natural numbers.

Proof.

$$\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0 \quad \dots (1)$$

$$\text{Putting } u(x) = x^{(n)}, \Delta u(x) = nhx^{(n-1)}, \Delta^2 u(x) = n(n-1)h^2 x^{(n-2)} \text{ in (1)}$$

Then,

$$n(n-1)h^2 x^{(n-2)} + nhx^{(n-1)}P(x) + x^{(n)}Q(x) = 0$$

$$n(n-1)h^2 + nh \frac{x^{(n-1)}}{x^{(n-2)}} P(x) + \frac{x^{(n)}}{x^{(n-2)}} Q(x) = 0$$

We know that

$$x^{(n)} = x(x-h)(x-2h)\dots(x-(n-3)h)(x-(n-2)h)(x-(n-1)h)$$

$$x^{(n-1)} = x(x-h)(x-2h)\dots(x-(n-3)h)(x-(n-2)h)$$

$$x^{(n-2)} = x(x-h)(x-2h)\dots(x-(n-3)h)$$

Then,

$$n(n-1)h^2 + nh(x-(n-2)h)P(x) + (x-(n-2)h)(x-(n-1)h)Q(x) = 0.$$

$$\text{Example 2. Solve } \Delta^2 y(x) - \frac{2h}{x}\Delta y(x) + \frac{2h^2}{x(x-h)}y(x) = \frac{-h}{(x+2h)(x+3h)}.$$

$$\text{Solution. } \Delta^2 y(x) - \frac{2h}{x}\Delta y(x) + \frac{2h^2}{x(x-h)}y(x) = \frac{-h}{(x+2h)(x+3h)}$$

Equate with $\Delta^2 y(x) + P(x)\Delta y(x) + Q(x)y(x) = R(x)$, we get

$$P(x) = \frac{-2h}{x}, Q(x) = \frac{2h^2}{x(x-h)}, R(x) = \frac{-h}{(x+2h)(x+3h)}$$

We know that $u(x) = x^{(n)}$ is a solution of equation

$$\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0,$$

$$\text{if } n(n-1)h^2 + nh(x-(n-2)h)P(x) + (x-(n-2)h)(x-(n-1)h)Q(x) = 0$$

where $n \in N$, N is the set of all natural numbers

For $n = 2$

$$2h^2 + 2hxP(x) + x(x-h)Q(x)$$

$$2h^2 + 2hx\left(-\frac{2h}{x}\right) + x(x-h)\frac{2h^2}{x(x-h)}$$

$$= 2h^2 - 4h^2 + 2h^2$$

$$= 0$$

Hence $u(x) = x^{(2)}$

$$\text{Now } I(x) = \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)}$$

$$\begin{aligned}
 &= \frac{4h(x+h) - \frac{2h}{x}(x+h)x}{(x+2h)(x+h)} \\
 &= \frac{4h(x+h) - 2h(x+h)}{(x+h)(x+2h)} \\
 &= \frac{2h}{x+2h} \\
 z(x) &= e^{\Delta^{-1} \log \left\{ \frac{1}{1-I(x)} \right\}} \\
 &= e^{\Delta^{-1} \log \left\{ \frac{1}{1 - \frac{2h}{x+2h}} \right\}} \\
 &= e^{\Delta^{-1} \log \left(\frac{x+2h}{x} \right)} \\
 &= e^{\Delta^{-1} \{ \log(x+2h) - \log(x+h) + \log(x+h) - \log x \}} \\
 &= e^{\Delta^{-1} \{ \Delta \log(x+h) + \Delta \log x \}} \\
 &= e^{\Delta^{-1} \Delta \log x(x+h)} \\
 &= x(x+h)
 \end{aligned}$$

$$\begin{aligned}
 \Delta v(x) &= \frac{1}{z(x)} \Delta^{-1} \left\{ z(x+h) \frac{R(x)}{u(x+2h)} \right\} + \frac{c_1}{z(x)} \\
 &= \frac{1}{x(x+h)} \Delta^{-1} \left\{ (x+h)(x+2h) \frac{-h}{(x+2h)(x+3h)} \frac{1}{(x+2h)(x+h)} \right\} + \frac{c_1}{x(x+h)} \\
 &= \frac{1}{x(x+h)} \Delta^{-1} \left\{ -h(x+h)^{(-2)} \right\} + \frac{c_1}{x(x+h)} \\
 &= \frac{1}{x(x+h)} \frac{(x+h)^{(-1)}}{(-1)h} (-h) + \frac{c_1}{x(x+h)} \\
 &= \frac{1}{x(x+h)(x+2h)} + \frac{c_1}{x(x+h)}
 \end{aligned}$$

$$\Delta v(x) = (x-h)^{(-3)} + c_1 (x-h)^{(-2)}$$

$$v(x) = \frac{(x-h)^{(-2)}}{-2h} + c_1 \frac{(x-h)^{(-1)}}{-h} + c_2$$

$$y(x) = u(x)v(x)$$

$$\begin{aligned}
 y(x) &= x^{(2)} \left\{ -\frac{1}{2h} \frac{1}{x(x+h)} - \frac{c_1}{h} \frac{1}{x} + c_2 \right\} \\
 &= -\frac{x-h}{2h(x+h)} - \frac{c_1}{h} (x-h) + c_2 x(x-h) \\
 y(x) &= -c_1 \left(\frac{x-h}{h} \right) + c_2 x(x-h) - \frac{x-h}{2h(x+h)}.
 \end{aligned}$$

Example 3. Solve $\Delta^2 y(x) - \frac{4h}{(x-h)} \Delta y(x) + \frac{6h^2}{(x-h)(x-2h)} y(x) = \frac{-hx}{(x+2h)(x+3h)}$.

Solution. $\Delta^2 y(x) - \frac{4h}{(x-h)} \Delta y(x) + \frac{6h^2}{(x-h)(x-2h)} y(x) = \frac{-hx}{(x+2h)(x+3h)}$

Equate with $\Delta^2 y(x) + P(x)\Delta y(x) + Q(x)y(x) = R(x)$, we get

$$P(x) = -\frac{4h}{x-h}, Q(x) = \frac{6h^2}{(x-h)(x-2h)}, R(x) = \frac{-hx}{(x+2h)(x+3h)}$$

We know that $u(x) = x^{(n)}$ is a solution of equation

$$\Delta^2 u(x) + P(x)\Delta u(x) + Q(x)u(x) = 0,$$

$$\text{if } n(n-1)h^2 + nh(x-(n-2)h)P(x) + (x-(n-2)h)(x-(n-1)h)Q(x) = 0$$

where $n \in N$, N is the set of all natural numbers

For $n = 3$

$$6h^2 + 3h(x-h)P(x) + (x-h)(x-2h)Q(x)$$

$$6h^2 + 3h(x-h)\left(-\frac{4h}{x-h}\right) + (x-h)(x-2h)\frac{6h^2}{(x-h)(x-2h)}$$

$$= 6h^2 - 12h^2 + 6h^2$$

$$= 12h^2 - 12h^2$$

$$= 0$$

Hence $u(x) = x^{(3)}$

$$\text{Now } I(x) = \frac{2\Delta u(x+h) + P(x)u(x+h)}{u(x+2h)}$$

$$= \frac{6h(x+h)x - \frac{4h}{(x-h)}(x+h)x(x-h)}{(x+2h)(x+h)x}$$

$$= \frac{6hx(x+h) - 4hx(x+h)}{x(x+h)(x+2h)}$$

$$= \frac{6h - 4h}{x+2h}$$

$$= \frac{2h}{x+2h}$$

$$z(x) = e^{\Delta^{-1} \log \left\{ \frac{1}{1-I(x)} \right\}}$$

$$= e^{\Delta^{-1} \log \left\{ \frac{1}{1-\frac{2h}{x+2h}} \right\}}$$

$$= e^{\Delta^{-1} \log \left(\frac{x+2h}{x} \right)}$$

$$= e^{\Delta^{-1} \{ \log(x+2h) - \log(x+h) + \log(x+h) - \log x \}}$$

$$= e^{\Delta^{-1} \{ \Delta \log(x+h) + \Delta \log x \}}$$

$$= e^{\Delta^{-1} \Delta \log x(x+h)}$$

$$= x(x+h)$$

$$\Delta v(x) = \frac{1}{z(x)} \Delta^{-1} \left\{ z(x+h) \frac{R(x)}{u(x+2h)} \right\} + \frac{c_1}{z(x)}$$

$$= \frac{1}{x(x+h)} \Delta^{-1} \left\{ (x+h)(x+2h) \frac{-hx}{(x+2h)(x+3h)} \frac{1}{(x+2h)(x+h)x} \right\} + \frac{c_1}{x(x+h)}$$

$$\begin{aligned}
 &= \frac{1}{x(x+h)} \Delta^{-1} \left\{ -h(x+h)^{(-2)} \right\} + \frac{c_1}{x(x+h)} \\
 &= \frac{1}{x(x+h)} \frac{(x+h)^{(-1)}}{(-1)h} (-h) + \frac{c_1}{x(x+h)} \\
 &= \frac{1}{x(x+h)(x+2h)} + \frac{c_1}{x(x+h)} \\
 \Delta v(x) &= (x-h)^{(-3)} + c_1 (x-h)^{(-2)} \\
 v(x) &= \frac{(x-h)^{(-2)}}{-2h} + c_1 \frac{(x-h)^{(-1)}}{-h} + c_2 \\
 y(x) &= u(x)v(x) \\
 y(x) &= x^{(3)} \left\{ -\frac{1}{2h} \frac{1}{x(x+h)} - \frac{c_1}{h} \frac{1}{x} + c_2 \right\} \\
 &= x(x-h)(x-2h) \left\{ -\frac{1}{2h} \frac{1}{x(x+h)} - \frac{c_1}{h} \frac{1}{x} + c_2 \right\} \\
 &= -\frac{1}{2h} \frac{(x-h)(x-2h)}{(x+h)} - \frac{c_1}{h} (x-h)(x-2h) + c_2 x(x-h)(x-2h) \\
 y(x) &= -\frac{c_1}{h} (x-h)(x-2h) + c_2 x(x-h)(x-2h) - \frac{1}{2h} \frac{(x-h)(x-2h)}{(x+h)}.
 \end{aligned}$$

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