



Dominant Eigenvector of Connected Graph

¹Dr. Maneesha Sakalle and ²Richa Jain

¹Professor, S.N. Govt. P.G. College, Khandwa (M.P.), INDIA

Email: maneeshasakalle@gmail.com

²Asst. Professor, Indore Institute of Science and Technology, Indore (M.P.), INDIA

Abstract: Let $G(V,E)$ be connected graph without any self loop. A lower bound and upper bound on the dominant eigenvalue will put upper bound and lower bound on ratio of entries of dominant eigenvector of connected graph G . In this paper we provide a proof for the upper bound and lower bound of ratio of component of dominant eigenvector of connected graph G .

Key Words: Spectral radius; Eigenvalue; Eigenvectors; Adjacency matrix; Dominant Eigenvalue.

I. INTRODUCTION

The distinctive and noteworthy strength of graph is its ability to deal with the (0,1) problems occurs in many problem. In applied jobs, decision makers give immanent judgements on the relative importance ratios; hence it seems to be plain to explore how the dominant eigenvalue and its associate eigenvector vary. F. Juhász[3] analysed the behaviour of the theta-function on random graphs and introduced eigenvalues in clustering. Wilf and Hoffman gave the eigenvalue bounds on the chromatic number at the end of sixties. An upper bound on the maximal entry in the principal eigenvector of a symmetric non negative matrix with zero diagonal entries is investigated by S. Zhong and Y. Hong [12]. A sharp upper bound on the maximal entry in the principal eigenvector of symmetric non negative matrix in terms of order, the spectral radius is studied by Kinker Ch. Das[10]. The rest of paper is organised as follows first we gave some basic terminology followed by some previous reviews, main result and references.

II. PRELIMINARIES

We shall use the standard terminology of graph theory as it is introduced in most of the text books on the theory of graph. We shall Let $G(V,E)$ be a connected graph with vertex set V of cardinality n and edge set E . Two vertices are said to be adjacent (or neighbours) if they are connected by an edge; the corresponding relation between vertices is called adjacency relation. The number of adjacent vertices of a vertex v_i , denoted by d_i is its degree. We denote the maximum and minimum degree of vertex by Δ and δ . The adjacency matrix A represents the adjacency relation between vertices. The entries a_{ij} of the adjacency matrix is 1 if vertices v_i and v_j are adjacent, and 0 otherwise. The characteristic polynomial $\det(A - \lambda I)$ of adjacency matrix A is called the characteristic polynomial of G , so the eigenvalues and eigenvectors of A are the eigenvalues and eigenvectors of G and usually denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. Since A is symmetric so the eigenvalues are real. We shall assume that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. The dominant eigenvalue of G is the largest eigenvalue λ_n of adjacency matrix A and we denote it by λ_{max} . The largest eigenvalue of A is called the spectral radius and denoted by γ .

If λ is an eigenvalue of G then non zero vector $X \in \mathbb{R}^n$ satisfying the relation $AX = \lambda X$ is called an eigenvector of G and the relation $AX = \lambda X$ can be represented as $X = (X_1, X_2, \dots, X_n)^T$, then for any vertex v we have $\lambda X_v = \sum_{v \sim u} X_u$, where the summation is over all neighbours u of v . The eigenvector of length 1 corresponding to λ_{max} is said to be dominant eigenvector and we denote it by X .

A. Section 2

Stevanović[1] prove the following result for upper bound of spectral radius of non-regular and connected graph.

Theorem 2.1: Let G be connected and non regular graph of order n then $\gamma < \Delta - \frac{1}{2n(n\Delta-1)\Delta^2}$.

Zhang[14] improves the result of upper bound for the spectral radius of non regular graph provided by Stevanović[1] then Papendieck and Recht proved that $X_{max} \leq 1/\sqrt{2}$.

Proposition 2.1: If G is a connected graph of order n then $X_{min} \leq (\Delta - \lambda_{max})(n\Delta - \sum_{i=1}^n d_i)^{-1} n^{1/2}$

Proof: By using the equality $\lambda_{\max} \sum_{j=1}^n X_j = \sum_{j=1}^n d_j X_j$

we have

$$(\Delta - \lambda_{\max}) \sum_{j=1}^n X_j = \sum_{j=1}^n (\Delta - d_j) X_j \geq X_{\min} \sum_{j=1}^n (\Delta - d_j) = X_{\min} (n\Delta - \sum_{j=1}^n d_j) \quad (1)$$

applying the inequality $\sum_{j=1}^n X_j \leq X_{\min} + \sqrt{(n-1)(1 - X_{\min}^2)}$ to eq. (1) we have

$$(\Delta - \lambda_{\max}) X_{\min} + \sqrt{(n-1)(1 - X_{\min}^2)} \geq X_{\min} (n\Delta - \sum_{j=1}^n d_j)$$

on solving we get, $X_{\min} \leq (\Delta - \lambda_{\max})(n\Delta - \sum_{i=1}^n d_i)^{-1} n^{1/2}$ as desired. $\square$

Proposition 2.2: If G is connected nonregular graph of order n and D is the diameter of G then

$$\gamma < \Delta + \frac{(\Delta - \delta)}{Dn\Delta - 2}$$

Proof: Let $X = (X_1, X_2, \dots, X_n)$ be the normalized eigenvector corresponding to $\lambda_{\max}$ then

$$\Delta - \gamma = \Delta \sum_{j=1}^n X_j^2 - X^T A X > \sum_{j=1}^n d_j X_j^2 - 2 \sum_{v_j \sim v_i} X_i X_j > \sum_{v_j \sim v_i} (X_j - X_i)^2$$

Let there exist a path $P = (v_a = v_{m_1}, v_{m_2}, \dots, v_{m_l} = v_b)$ of length l then by using Cauchy Schwarz inequality, we

$$\text{have } \Delta - \gamma > \sum_{v_j \sim v_i} (X_j - X_i)^2 > \sum_{j=1}^l \frac{(X_{m_{i+1}} - X_{m_i})^2}{l} \geq \frac{(X_a - X_b)^2}{l}$$

$$\text{Now we know that } \frac{X_a}{X_b} \geq \sqrt{\frac{\Delta}{\delta}}$$

$$\text{So, } X_a - X_b \geq X_a - X_a \sqrt{\frac{\delta}{\Delta}} = \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta}} X_a$$

Moreover, since $X_1, X_2, \dots, X_n$ are normalized eigenvector so $\sum_{j=1}^n X_j^2 = 1$ will implies $X_a > \sqrt{\frac{1}{n}}$

$$\text{Hence, } \Delta - \gamma > \frac{(X_a - X_b)^2}{l} \geq \frac{(\Delta + \delta - 2\sqrt{\Delta\delta})}{\Delta n} \text{ which yields } \gamma < \Delta + \frac{(\Delta - \delta)}{Dn\Delta - 2}. \quad \square$$

B. Section 3

One of the main goals in connected graph is to deduce the principal properties and structure of a connected graph from its graph spectrum. Also in connected graph the multiplicity of eigenvalue 0 is 1, since any harmonic eigen function with eigenvalue 0 assumes the same value at each vertex.

If X_i and X_j are the components of X , Define

$$\alpha(G) = \max_{1 \leq i \leq n, 1 \leq j \leq n} \frac{X_i}{X_j}$$

It is well known that if G is regular and connected graph then $\alpha(G) = 1$. Schneider has been proved that if G is connected graph then $\alpha(G) \leq \lambda_{n-1}$

Theorem 3.1: If G is connected graph of order n with dominant eigenvalue $\lambda_{\max} > 2$ and dominant eigenvector X .

Let l be the length of shortest path between the vertex on which X is minimum and maximum then $\frac{(a^{2l+2}-1)}{a^{l+1}}$ $(\lambda_{\max}^2 - 4)^{-1/2}$ is an upper bound of $\alpha(G)$, where $a = \frac{1}{2}(\lambda_{\max} + \sqrt{\lambda_{\max}^2 - 4})$

And also $\alpha(G) = \frac{(a^{2l+2}-1)}{a^{l+1}} (\lambda_{\max}^2 - 4)^{-1/2}$ if and only if there exist an induced path $P = \{v_{\min}, v_1, v_2, \dots, v_{l-2}, v_{l-1}, v_{\max}\}$ of length l such that $d(v_{\min})$ and $d(v_{\max})$ is 1 or 3 or more and $d(v_1) = d(v_2) = \dots = d(v_{l-2}) = 2$.

Before proving this theorem let us remark some useful observations.

Theorem (Perron-Frobenius) 3.2: If an $n \times n$ matrix has nonnegative entries then it has a nonnegative real eigenvalue λ , which has maximum absolute value among all eigenvalues. This eigenvalue λ has a nonnegative real eigenvector. If, in addition the matrix has no block triangular decomposition (i.e., it does not contain a $k \times (n-k)$ block of 0's disjoint from the diagonal), then λ has multiplicity 1.

The Perron Frobenius Theorem immediately implies if G is connected then $\lambda_{\max}$ has multiplicity 1. If γ is the spectral radius and $\lambda X_v = \sum_{v \sim u} X_u$ then the following result is obvious

$$\frac{X_i}{X_j} \leq \frac{\gamma^r - \gamma^{-r}}{\gamma - \gamma^{-1}}, \text{ where } j < i < r < n \quad (2)$$

Proof of theorem 3.1: Let X be the dominant eigenvector of connected graph G . If G is regular and connected graph then $l=0$ will implies upper bound equals 1.

If G is irregular and connected graph then let $X_{\min}$ and $X_{\max}$ be the smallest and largest entries of X and also let P_{l+1} be the length of path connecting vertices v_1 and v_{l+1} . Since $\lambda_{\max} > 2$ then by eq. (2) it follows that

$$X_{l+1} \leq \frac{(a^{2l+2}-1)}{a^{l+1}} (\lambda_{\max}^2 - 4)^{-1/2} X_1, \text{ implies } X_{l+1}/X_1 \leq \frac{(a^{2l+2}-1)}{a^{l+1}} (\lambda_{\max}^2 - 4)^{-1/2}$$

$$\text{implies } \alpha(G) \leq \frac{(a^{2l+2}-1)}{a^{l+1}} (\lambda_{\max}^2 - 4)^{-1/2} \quad (3)$$

If $l > 0$ and equality holds. Suppose P_k be the path on first k vertices $v_1, v_2, \dots, v_k$ on a subgraph of G then by eq. (2) vertex v_1 has degree 1 and vertices $v_2, v_3, \dots, v_{k-1}$ all have degree 2. Vertex v_k cannot have degree 1 otherwise, $X_{k-1} = \lambda_{\max} X_k > X_k$, a contradiction. Suppose v_k has one more adjacent vertex in form of X_{k+1} . Then $\lambda_{\max} X_k = X_{k+1} + X_k < 2X_k$ will imply $\lambda_{\max} < 2$, a contradiction. Thus $d(X_k) = 3$. Conversely, if such a path exists and the endpoints index utmost entries then inequality (3) converts into equality and we have desired result. $\square$
Similarly, we can immediately prove the lower bound of $\alpha(G)$.

Theorem (Xiong-Dong Zhang) 3.3: Let G be a simple connected and non-regular graph of order n with degree sequences $(d_1, d_2, \dots, d_n)$. Denote by $\bar{d}$ the average degree of G . If $X = (X_1, X_2, \dots, X_n)^T$ is a dominant eigenvector of G then

$$\alpha(G) = \max_{1 \leq i, j \leq n} \frac{X_i}{X_j} \geq \frac{(\Delta - \bar{d})(\gamma - \delta)}{(\Delta - \gamma)(\bar{d} - \delta)},$$

with equality if and only if there exists a partition of vertex set $V = V_1 \cup V_2$ and $0 \leq k < \Delta$ and $0 \leq l < \delta$ such that each vertex in v_1 is adjacent to v_k and $\Delta - k$ vertices in v_1 and v_2 respectively, and each vertex in v_2 is adjacent to l and $\delta - l$ vertices in v_2 and v_1 respectively.

Theorem 3.4: If G is connected graph without any self loop and A be its non negative adjacency matrix and Let

$$M = \{v_m : d_m > \gamma\} \text{ and } N = \{v_n : d_n < \gamma\} \text{ then } \left\{ \frac{\Delta + \sum_{v_k \in M \setminus \{v_m\}} \frac{a_{mk}(d_k - \gamma)}{\gamma}}{\gamma} \right\} \text{ and } \left\{ \frac{\gamma}{\delta - \sum_{v_k \in N \setminus \{v_n\}} \frac{a_{nk}(\gamma - d_k)}{\gamma}} \right\} \text{ are the lower bounds}$$

of $\alpha(G)$.

where $v_m \in M$ such that $d(v_m) = \Delta$ and $v_n \in N$ such that $d(v_n) = \delta$ and a_{mk} is the entry of A with respect to v_k and v_m .

Proof:- By simple calculations we can prove the above result. For each k , we have

$$\gamma X_k = \sum_{i \neq k} a_{ki} X_i \geq d_k X_{\min} > 0, \text{ since } A \text{ is non-negative will imply } \frac{X_k}{X_{\min}} \geq \frac{d_k}{\gamma}$$

$$\text{Now } \gamma X_j = \sum_{v_k \in N \setminus \{v_n\}} a_{jk} X_k + \sum_{v_m \in M} a_{jm} X_m$$

Since $d_j = \Delta$ will imply

$$\gamma X_j \geq X_{\min} \sum_{v_k \in N \setminus \{v_n\}} a_{jk} \frac{d_k}{\gamma} + (\Delta - \sum_{v_m \in M} a_{jm}) X_{\min}, \text{ implies}$$

$$\gamma X_j \geq X_{\min} \sum_{v_k \in N \setminus \{v_n\}} a_{jk} \frac{(d_k - \gamma + \gamma)}{\gamma} + (\Delta - \sum_{v_m \in M} a_{jm}) X_{\min}, \text{ leads to}$$

$$\gamma X_j \geq X_{\min} \left(\sum_{v_k \in N \setminus \{v_n\}} a_{jk} \frac{(d_k - \gamma + \gamma)}{\gamma} + \Delta - \sum_{v_m \in M} a_{jm} \right), \text{ implies}$$

$$\alpha(G) = \frac{X_j}{X_{\min}} \geq \left\{ \frac{\Delta + \sum_{v_k \in M \setminus \{v_m\}} \frac{a_{mk}(d_k - \gamma)}{\gamma}}{\gamma} \right\} \quad (4)$$

Now

$$0 < \gamma X_k = \sum_{i \neq k} a_{ki} X_i \leq d_k X_{\max}$$

$$\text{will imply } \frac{X_{\max}}{X_k} \geq \frac{\gamma}{d_k} > 0$$

$$\text{Now } \gamma X_i = \sum_{v_k \in M \setminus \{v_m\}} a_{ik} X_k + \sum_{v_n \in N} a_{in} X_n$$

Since $d_i = \delta$ will imply

$$\gamma X_i \leq X_{\max} \sum_{v_k \in M \setminus \{v_m\}} a_{ik} \frac{d_k}{\gamma} + (\delta - \sum_{v_n \in N} a_{in}) X_{\max}, \text{ implies}$$

$$\gamma X_i \leq X_{\max} \sum_{v_k \in M \setminus \{v_m\}} a_{ik} \frac{(d_k - \gamma + \gamma)}{\gamma} + (\delta - \sum_{v_n \in N} a_{in}) X_{\max}, \text{ implies}$$

$$\alpha(G) = \frac{X_i}{X_{\max}} \geq \left\{ \frac{\gamma}{\delta - \sum_{v_k \in N \setminus \{v_n\}} \frac{a_{nk}(\gamma - d_k)}{\gamma}} \right\} \quad \square$$

In real world problem the appearance of smallest eigenvalue is not crucial and what we need to work on largest eigenvalue. Another consequence of largest Eigenvalue is that it has application to web search algorithms that are based on adjacency matrices of hyperlinks between web pages. It is of interest to see what further possible upper bound and lower bound for $\alpha(G)$.

References

- [1] D. Stevanović, "The Largest Eigenvalue of nonregular Graph", J. Combin. Theory, pp.143-146,[2004], vol. 91
- [2] F.Canto- Martin, H.Hedenmalm, A.Montes-Rodriguez, "Perron-Frobenius Operators and the Klein-Gordon equation", European Mathematical Society, pp.31-66, [2014], vol.16/1
- [3] F. Juhász, "The Asymptotic behaviour of Lavász ϑ function for Random Graphs", Combinatorica, pp. 153-155[1982],vol. 2
- [4] Francis K. Bell, Dragoš Cvetković, Peter Rowlinson, Slobodan K. Simić, "Graphs for which the least Eigenvalue is minimal, II", Linear Algebra and its Applications, pp. 2168-2179 [2008] vol. 429
- [5] G.N. De Oliveira, "On the character vectors of a matrix", Linear Algebra and its Applications, pp. 189-196[1972], vol.5
- [6] H. Minc, "On the Maximal Eigenvector of a positive matrix". SIAM Journal on Numerical Analysis-7, pp. [424-427]
- [7] Houqing Zhou, "The Main Eigenvalues of the Seidel Matrix", Mathematica Moravica, pp. 111-116 [2008] vol. 12
- [8] Hong Yuan, "The k th largest Eigenvalue of a Tree", Linear Algebra Application, pp. 151-155 [1986], vol. 73

- [9] H. Schneider, "*Note on the fundamental theorem on irreducible non-negative matrices*", Proceedings of the Edinburg Mathematical Society, pp. 127-130,[1958/1959], vol.11
- [10] Kinkar Ch. Das, "*A Sharp Upper bound on the maximal entry in the Principal Eigenvector of Symmetric nonnegative Matrix*", Linear Algebra and its Applications, pp. 1340-1350 [2009], vol. 431
- [11] Sebastien M. Cloaba, David A. Gregory, "*Principal Eigenvectors of Irregular Graphs*", Electronic Journal of Linear Algebra ISSN 1081-3801 A Publication of the Linear Algebra Society, pp. 366-379[2007] , vol. 16
- [12] S. Zhao, Y. Hong, "*On the bounds of maximal entries in the principal eigenvector of symmetric non-negative matrix*" , Linear algebra Application ,pp. 245-252, [2002] , vol. 340
- [13] Tracy Powell , "*Bounds on the Ratio of Eigenvalues*", Department of Mathematics Harvey Mudd college [2005]
- [14] V. Nikiforov "*Walks and the Spectral radius of Graphs*", Linear Algebra and its Applications, pp. 257-268 [2006], vol.418
- [15] Xiong-Dong Zhang, "*Eigenvectors and eigenvalues of non-regular graphs*", Linear Algebra and its Applications, pp.79-86[2005] vol. 409
- [16] Zoran Stanić, "*Some Star Complements for the Second Largest Eigenvalue of a Graph*" , Ars Mathematica Contemporanea-1, pp. 126-136 [2008]

Acknowledgements

We thank an anonymous referee for some useful comments that helped us to improve the readability of our paper.