



A Computational Study of the Non-Dimensional Two Lane Traffic Flow Model with Two Sided Boundary Conditions

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Abstract: This paper focuses on the computational study of the non-dimensional form of the two lane traffic flow model with two sided boundary conditions based on a linear density-velocity relationship. By using some dimensionless parameters we transform the two lane traffic flow model in the dimensionless form. In order to compute the numerical solution of the model as an initial boundary value problem (IBVP) with two sided boundary conditions, we describe the discretization of the model by using finite difference method which leads to the Lax-Friedrichs scheme. A numerical simulation of 2.5km highway of two lanes is performed for 1.5 minutes using the numerical scheme based on artificially generated initial and boundary data.

Keywords: Non-Dimensional form; Two Lane Traffic Flow Model; Two Sided Boundary Conditions; Numerical Simulation.

I. Introduction

Congestion of vehicular traffic within urban areas is a problem experienced worldwide. It has adverse effects on people quality of life due to delays, accidents and environmental pollution. One way of eliminating the problem is to increase the capacity of existing roadways by addition of lanes. However, this is greatly hampered by lack of space, resources or due to environmental issues and sometimes politics. This leaves the relevant authorities with one major option, that of enhancing the utilization of existing infrastructure by employing better traffic management and operations strategies. For effective traffic management and control, proper understanding of traffic congestion is needed. To achieve the latter, spatiotemporal behavior of empirical traffic congested patterns should be studied closely. This is because traffic congestion is observed to take place in space and time in form of spatiotemporal congested traffic patterns that propagate within roadways.

Traffic flow is the study of movement of individual drivers and vehicles between two points and the interactions they make with one another. With the rise of the automobile as one of the most important means of transport in the modern world, traffic problems have become one of the main societal and economical issues in almost all industrialized countries. Consequently, traffic issues often play an important role in both policy and research. In the 1950s scientists from many disciplines started to model the movement of traffic with the aim to improve the traffic operations. One of the pioneering contributions to traffic flow theory was the model of Lighthill and Whitham (1955). They proposed a macroscopic traffic mode akin to the dynamics of particles in a fluid. To date, many efforts have been undertaken to improve the original model of Lighthill and Whitham, modeling traffic flow as a continuum. The continuum traffic flow model pertaining to the modeling of discontinuities of traffic flow at bottlenecks on multi-lane freeway and in urban networks developed in [1]. The dimensionless form of the macroscopic single lane traffic flow model is presented in [2]. The numerical experiments of the scaled traffic flow model are also shown here. Some theoretical results are also established in [2].

The fluid dynamic traffic flow model as an IBVP with two sided boundary condition and a new version of the Lax-Friedrichs scheme for the fluid dynamic traffic flow model is also presented in [3]. The traffic flow in multilane is studied in [4]. Multilane traffic flow model is presented and some numerical experiments for multilane traffic flow model are also performed in [4]. The authors present the macroscopic traffic flow model appended with nonlinear density velocity function. They perform numerical experiments in order to verify some qualitative traffic flow behavior for traffic flow parameters. In [5], [6], the finite difference scheme for fluid dynamic single lane traffic flow model is studied. They also verify some well-known traffic flow behaviors

using such numerical scheme. A new version of the Lax-Friedrichs scheme for the fluid dynamic traffic flow model is also presented in [7].

In this article, we choose a macroscopic two lane traffic flow model by approximating a linear density-velocity relationship. By using some dimensionless parameters from the two lane traffic flow model we get an IBVP from a particular two lane traffic flow model. By using Lax-Friedrichs scheme for our considered model we find the numerical simulation of 2.5 km highway of two lanes for 1.5 minutes by using the numerical scheme based on artificially generated by two sided boundary data. We build up a code in MatLab programming language of scientific computing for the Lax-Friedrichs scheme. We present the density profile as well as computed velocity and flux profiles of our considered model which is scaled (non-dimensional) model without transformation back. We present the numerical solution of the scaled two lane traffic flow model for linear velocity-density relationships by using the Lax-Friedrichs scheme.

II. Macroscopic Multilane Traffic Flow Model

Macroscopic (continuum) traffic flow models vehicle transition rate from lane j to lane k , with $|j - k| = 1$ deal with traffic flow in terms of aggregate variables as a function of location and time. They describe the dynamics of traffic density $\rho(x, t)$, mean speed $V(x, t)$ and/or flow rate $q(x, t)$. The first equation of a macroscopic model is the so-called conservation equation, which describes that the number of vehicles flowing into a cell $[x, x + dx]$ during time interval $[t, t + dt]$ is equal to the number of vehicles remaining in that cell and the number of vehicles flowing out from that cell. The conservation equation is formulated as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (1)$$

We assume a highway with N lanes. They are numbered by $\alpha = 1, \dots, N$. The maximal velocity is denoted by w ; i.e., the velocities range between 0 and w . The single car distribution function denoted by $f_\alpha(x, v)$ describes the number of cars at x with velocity v on lane α . The kinetic multilane traffic flow model is given by

$$\partial_t f_\alpha + v \partial_x f_\alpha = C_\alpha^+(f_1, \dots, f_N) \quad (2)$$

In our research, we focus on the multilane traffic model (2). Using the definition of Kronecker symbol the multilane model (2) can be written in generalized form as follows from [7]:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_j}{\partial t} + \frac{\partial q_j}{\partial x} &= \frac{\rho_{j-1}}{T_{j-1}^j} - \frac{\rho_j}{T_j^{j-1}} + \frac{\rho_{j+1}}{T_{j+1}^j} - \frac{\rho_j}{T_j^{j+1}} \\ \frac{\partial \rho_N}{\partial t} + \frac{\partial q_N}{\partial x} &= \frac{\rho_{N-1}}{T_{N-1}^N} - \frac{\rho_N}{T_N^{N-1}} \end{aligned} \quad (3)$$

Here, the subscripts $1, j = 2, \dots, N - 1$ and N refer to the lane numbers. The quantities ρ_j and $q_j = \rho_j v_j$ are the vehicle density and the vehicle flux in the j -th lane respectively whereas v_j is the vehicle velocity at the j -th lane for $j = 1, \dots, N$; at last $T_j^k = T_j^k(\rho_j, \rho_k)$ is the transition rate from j -th lane to k -th lane. In particular, we choose macroscopic multilane traffic flow model (3) for two lanes that is for $j = 1, 2$ ($N = 2$). In that case the multilane traffic model (3) takes the following simplified form:

$$\left. \begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial q_2}{\partial x} &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \end{aligned} \right\} \quad (4)$$

III. Density-Velocity Relationship

The macroscopic multilane traffic flow model for two lanes (4) is approximated by the Greenshield linear density-velocity relationship as follows:

$$V(\rho) = V_{max} \left(1 - \frac{\rho}{\rho_{max}} \right)$$

Where, V_{max} is the maximum velocity and ρ_{max} is the maximum density which is based on bumper to bumper traffic. By using the Density-Velocity relationship we get from the equations (4), which takes the following form.

$$\left. \begin{aligned} \frac{\partial \rho_1}{\partial t} + \left[V_1^{max} \rho_1 \left(1 - \frac{\rho_1}{\rho_1^{max}} \right) \right]_x &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^1} \\ \frac{\partial \rho_2}{\partial t} + \left[V_2^{max} \rho_2 \left(1 - \frac{\rho_2}{\rho_2^{max}} \right) \right]_x &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^2} \end{aligned} \right\} \quad (5)$$

IV. Non-Dimensional Form of the Two Lane Traffic Flow Model

The dimensionless form of the two lane traffic flow model presented below. Now equations in (5) can be simplified by bringing it in a dimensionless form. Let L and τ be a typical length and time respectively, such that $\frac{L}{\tau} = V_{max}$. Introducing the following dimensionless parameters to make the two lane traffic flow model non-dimensional.

$$x_s = \frac{x}{L}, \quad t_s = \frac{t}{\tau}, \quad u_1 = 1 - \frac{2\rho_1}{\rho_1^{max}}, \quad u_2 = 1 - \frac{2\rho_2}{\rho_2^{max}}$$

By using these dimensionless parameters, from (5) we get

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} &= \frac{1}{\tau} \frac{\partial}{\partial t_s} \left[\frac{\rho_1^{max}}{2} (1 - u_1) \right] = -\frac{\rho_1^{max}}{2\tau} \frac{\partial u_1}{\partial t_s} \\ \left[V_1^{max} \rho_1 \left(1 - \frac{\rho_1}{\rho_1^{max}} \right) \right]_x &= \frac{1}{L} \frac{\partial}{\partial x_s} \left[V_1^{max} \frac{\rho_1^{max}}{2} (1 - u_1) \frac{1}{2} (1 + u_1) \right] = -\frac{\rho_1^{max}}{2\tau} \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) \end{aligned}$$

Similarly we write,

$$\begin{aligned} \frac{\partial \rho_2}{\partial t} &= \frac{1}{\tau} \frac{\partial}{\partial t_s} \left[\frac{\rho_2^{max}}{2} (1 - u_2) \right] = -\frac{\rho_2^{max}}{2\tau} \frac{\partial u_2}{\partial t_s} \\ \left[V_2^{max} \rho_2 \left(1 - \frac{\rho_2}{\rho_2^{max}} \right) \right]_x &= -\frac{\rho_2^{max}}{2\tau} \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) \end{aligned}$$

Hence we can write equations in (5) as (with (x, t) instead of (x_s, t_s)) the equations become,

$$\begin{aligned} \frac{\partial u_1}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) &= \frac{\tau}{T_2^1} (u_2 - 1) - \frac{\tau}{T_1^1} (u_1 - 1) \\ \frac{\partial u_2}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) &= \frac{\tau}{T_1^2} (u_1 - 1) - \frac{\tau}{T_2^2} (u_2 - 1) \end{aligned}$$

For $x_s \in \mathcal{R}, t_s > 0$, Finally, the equations can be written as an initial boundary value problem (IBVP) which is presented below.

$$\begin{aligned} \frac{\partial u_1}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) &= \frac{\tau}{T_2^1} (u_2 - 1) - \frac{\tau}{T_1^1} (u_1 - 1) \\ \frac{\partial u_2}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) &= \frac{\tau}{T_1^2} (u_1 - 1) - \frac{\tau}{T_2^2} (u_2 - 1) \\ u_1(x, 0) &= u_1^0(x); \quad u_2(x, 0) = u_2^0(x) \\ u_1(a, t) &= u_1^a(t); \quad u_2(a, t) = u_2^a(t) \end{aligned} \quad (6)$$

It is observed that the equations in the above IBVP (6) are analogous to the Forced Burgers' Equation.

V. Lax-Friedrichs Scheme for the Two Lane Model

For Lax-Friedrichs scheme, we have to make the model as an IBVP with two sided boundary conditions which takes the following form

$$\begin{aligned} \frac{\partial u_1}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) &= \frac{\tau}{T_2^1} (u_2 - 1) - \frac{\tau}{T_1^1} (u_1 - 1) \\ \frac{\partial u_2}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) &= \frac{\tau}{T_1^2} (u_1 - 1) - \frac{\tau}{T_2^2} (u_2 - 1) \\ u_1(x, 0) &= u_1^0(x); \quad u_2(x, 0) = u_2^0(x) \\ u_1(a, t) &= u_1^a(t); \quad u_1(b, t) = u_1^b(t) \\ u_2(a, t) &= u_2^a(t); \quad u_2(b, t) = u_2^b(t) \end{aligned} \quad (7)$$

In order to develop the numerical scheme, we discretize the temporal derivatives $\frac{\partial u_1}{\partial t_s}$ and $\frac{\partial u_2}{\partial t_s}$ by the forward difference in time and the spatial derivatives $\frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right)$ and $\frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right)$ by the central differences in space as presented below.

$$\left. \begin{aligned} \frac{\partial u_1}{\partial t_s} &\approx \frac{u_{1_i}^{n+1} - u_{1_i}^n}{\Delta t} \\ \frac{\partial u_2}{\partial t_s} &\approx \frac{u_{2_i}^{n+1} - u_{2_i}^n}{\Delta t} \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} \frac{\partial}{\partial x} \left(\frac{u_1^2}{2} \right) &\approx \frac{(u_{1_{i+1}}^n)^2 - (u_{1_{i-1}}^n)^2}{2\Delta x} \\ \frac{\partial}{\partial x} \left(\frac{u_2^2}{2} \right) &\approx \frac{(u_{2_{i+1}}^n)^2 - (u_{2_{i-1}}^n)^2}{2\Delta x} \end{aligned} \right\} \quad (9)$$

We assume the uniform grid spacing with step size k & h for time and space respectively $t^{n+1} = t^n + k$ & $x_{i+1} = x_i + h$. We write $(u_1)_i^n$ for $u_1(t, x)$ and $(u_2)_i^n$ for $u_2(t, x)$ in equations of (7). We also write $(u_1)_i^n$ for $u_1(t, x)$ and $(u_2)_i^n$ for $u_2(t, x)$ in the equations of (7).

$$u_{1_i}^{n+1} = \frac{1}{2} [u_{1_{i+1}}^n + u_{1_{i-1}}^n] - \frac{\Delta t}{2\Delta x} [(u_{1_{i+1}}^n)^2 - (u_{1_{i-1}}^n)^2] + \frac{\tau \Delta t}{2T_1^2} (u_{2_{i+1}}^n + u_{2_{i-1}}^n) - \frac{\tau \Delta t}{2T_1^2} (u_{1_{i+1}}^n + u_{1_{i-1}}^n) + \left(\frac{\tau}{T_1^2} - \frac{\tau}{T_2^2} \right) \quad (10)$$

Similarly,

$$u_{2_i}^{n+1} = \frac{1}{2} [u_{2_{i+1}}^n + u_{2_{i-1}}^n] - \frac{\Delta t}{2\Delta x} [(u_{2_{i+1}}^n)^2 - (u_{2_{i-1}}^n)^2] + \frac{\tau \Delta t}{2T_2^2} (u_{1_{i+1}}^n + u_{1_{i-1}}^n) - \frac{\tau \Delta t}{2T_1^2} (u_{2_{i+1}}^n + u_{2_{i-1}}^n) + \left(\frac{\tau}{T_2^2} - \frac{\tau}{T_1^2} \right) \Delta t \quad (11)$$

The above difference equations (10) and (11) are the Lax-Friedrichs scheme for the IBVP (7).

VI. Numerical Simulation

In this section, we investigate numerically the ability of the non-dimensional traffic flow model which we described in the previous sections. To this end, we simulate two lane traffic flow. The simulation involves obtaining a numerical solution of the model (6). We use Lax-Friedrichs scheme that is described above to compute numerical solution of the model (6) as an IBVP. In order to implement the numerical scheme, we develop an algorithm and build up a code in MatLab language of scientific computing. Finally, we visualize the outcome of the two lanes traffic flow simulation with respect to a variety of traffic flow parameters.

Here we choose the maximum velocity, $V_1^{max} = V_2^{max} = 100$ km/hour. We also consider the maximum density, $\rho_1^{max} = \rho_2^{max} = 175$ /km, Transition rates, $T_2^1 = 20\%$, $T_1^2 = 10\%$ and perform numerical simulation for 90 seconds in $\delta t = 1500$ time steps for a highway of two lanes of 2.5 km. We present the computed density profile using Lax-Friedrichs scheme in the Fig. 1. The behavior of car density with respect to highway in km for 1.5 minutes is observed in the following figure.

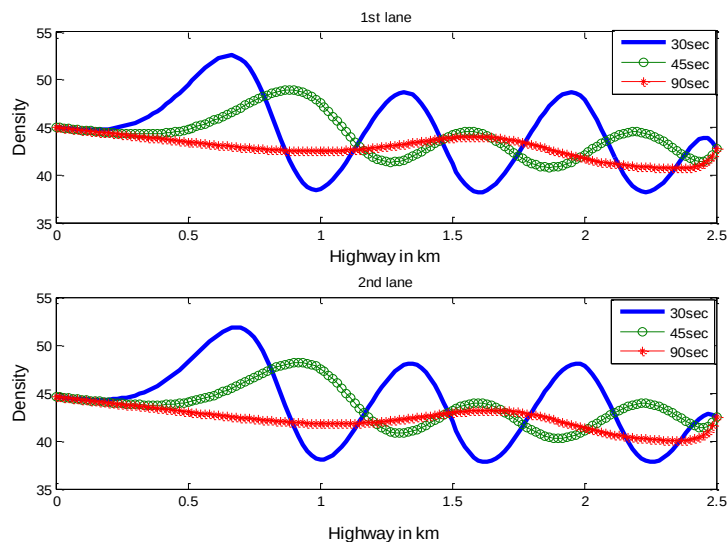


Figure 1 Computed Density Profile

Here we observed that as time goes on the traffic wave is moving forward with reducing wave height. By using the Lax-Friedrichs scheme we get the density for both the lane of traffic flow model. From our considered density-velocity relationship we can also compute the velocity. The computed velocity is presented with respect to the highway in the following Fig. 2.

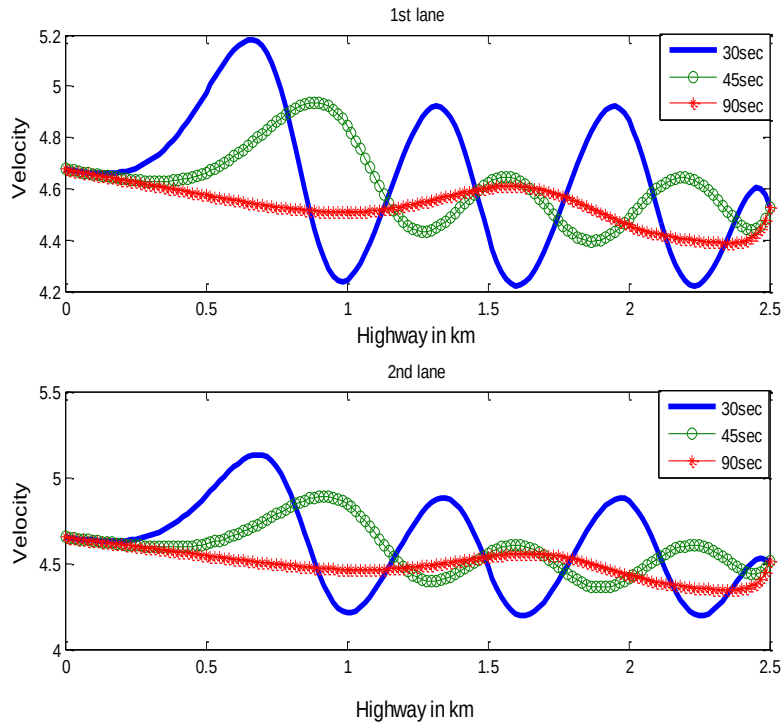


Figure 2 Computed Velocity Profile

At this stage, we are known about the density and velocity of two lanes traffic flow for a certain highway. So now we are capable to determine the flux of multilane traffic with the support of the relation, $q = \rho V$. We compute the flux which is presented with respect to the distance in the Fig. 3 for three different time steps as follows for both the lanes.

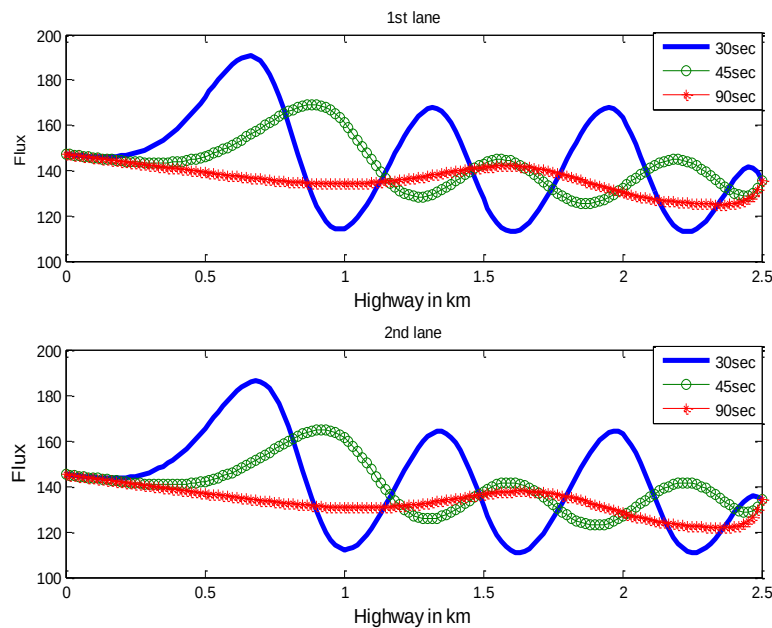


Figure 3 Computed Flux Profile

We compute the linear density velocity relation by using the macroscopic multilane traffic flow model for two lanes is approximated by the Greensfield density-velocity relationship $V(\rho) = V_{max} \left(1 - \frac{\rho}{\rho_{max}}\right)$ which depicted as in the Fig. 4. From the figure 4 it is also seen that the density and velocity relationship is linear which agrees accurately with our assumptions.

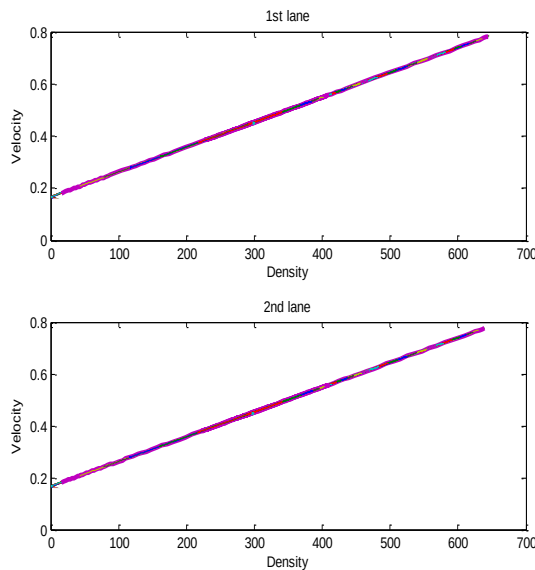


Figure 4 Density-Velocity Relationship

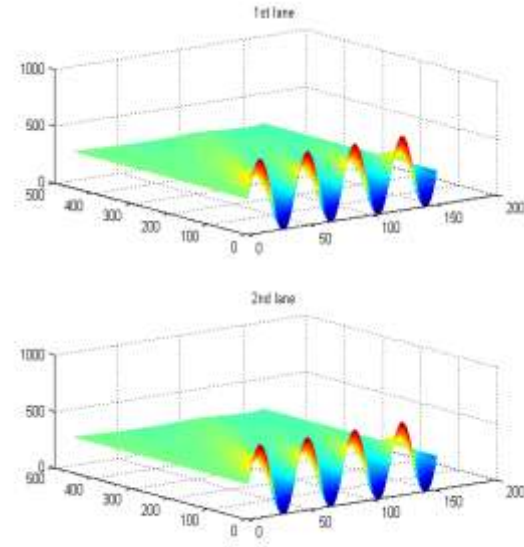


Figure 5 3D Visualization of Density

In the Fig. 5, the solution surface of the scaled two lane traffic flow model is depicted with respect to the distance and time using the Lax-Friedrichs scheme. The following figure is also a 3D visualization of the density of the considered model.

VI. Concluding Remarks

We have considered a macroscopic two lane traffic flow model by approximating a linear density-velocity relationship. Two lane traffic flow model has been made non-dimensional by introducing some dimensionless parameters. In order to get discrete version of the model, the traffic flow model has been discretized by using finite difference formulae which leads to the Lax-Friedrichs scheme. The numerical simulation of 2.5 km highway of two lanes has been performed for 1.5 minutes using the numerical scheme based on artificially generated initial and boundary data with respect to the dimensionless parameters. Since the non-dimensional model is simpler than the dimensional model, therefore we have observed much more computational efficiency for the non-dimensional model and that has been reduced the computational complexity as well.

VII. References

- [1] Dong NGODUY, Macroscopic Discontinuity Modeling for Multiclass Multilane Traffic Flow Operations, Master of Science in Engineering Thesis, Delft University of Technology, Netherland, 2006
- [2] Axel Klar & Raimund Wegner, A Hierarchy of Models for Multilane Vehicular Traffic I- Modeling, SIAM J. Appl. Math, Vol.59(3), pp.983-1001, 2010.
- [3] M O Gani, M M Hossain and L S Andallah, A finite difference scheme for a fluid dynamic traffic flow model appended with two-point boundary condition, GANIT: Journal of Bangladesh Mathematical Society, Vol. 31, pp. 43-52, 2011
- [4] Rinaldo M. Colombo, Andrea Corli, On Multilane Traffic Flow, Communications to SIAMI Congress, ISSN 1827-9015, Vol. 1, 2006
- [5] L S Andallah and M H Kabir, Simulation of Traffic Flow Problem, Research Project Report 2010-11, Faculty of Mathematical and Physical Science, Jahangirnagar University, Dhaka
- [6] M H Kabir, M O Gani and L S Andallah, Numerical Simulation of Mathematical Model of Traffic flow based on a Nonlinear Density-Velocity Function, Journal of Bangladesh Academy of Sciences, Vol. 34(1) pp. 15-22, 2010
- [7] M. H. Lighthill, and G. B. Whitham, On kinematic waves II: a theory of traffic flow on long crowded roads, Proceedings of the Royal Society of London series A, 229, 317-345, 1955
- [8] Atia Afroz, Numerical Simulation of a Dimensionless Form of Multilane Traffic Flow Model, M. Phil. Thesis 2013, Department of Mathematics, Jahangirnagar University, Dhaka, Bangladesh, 2013
- [9] Randall J. LeVeque, Numerical Methods for Conservation Laws, Basel; Boston; Berlin: Birkhauser, 1992

- [10] Muhammad Humayun Kabir, Numerical Simulation of Multilane Traffic Flow Model, M. Phil. Thesis 2012, Department of Mathematics, Jahangirnagar University, Dhaka, Bangladesh.
- [11] M H Kabir and L S Andallah, Numerical Solution of a Multilane Traffic Flow Model, Ganit: Journal of Bangladesh Society, Vol. 33, 2013, pp. 25-32
- [12] Jonathan A. Ward, Heterogeneity, Lane-Changing and Instability in Traffic: A Mathematical Ph. D. Thesis 2009, Department of Engineering Mathematics, University of Bristol, UK.
- [13] Michael Herty, Lorenzo Pareschi and Mohammed Sead, Enskog-Like Discrete Velocity Models for Vehicular Traffic Flow, NETWORKS AND HETEROGENEOUS MEDIA, American Institute of Mathematical Sciences, Vol. 2(3), 2007
- [14] Haijun Huang, Tieqiao Tang and Ziyao Gao, Continuum Modeling For Two-Lane Traffic Flow, Acta Mech Sinica 22: 131–137, 2006
- [15] J Sewall, D Wilkie, P Merrell and M C Lin, Continuum Traffic Simulation, EUROGRAPHICS 2010, Volume 29 (2)
- [16] Boris N. Chetverushkin *et al.*, A Two-Dimensional Macroscopic Model of Traffic Flows based on KCFD- Schemes, European Conference on Computational Fluid Dynamics, ECCOMAS CFD, TU Delft, The Netherlands, 2006
- [17] Emiliano Cristiani, Corrado de Fabritiis, Benedetto Piccoli, A Fluid Dynamic Approach for Traffic Forecast from Mobile Sensor Data, Communications in Applied and Industrial Mathematics, DOI: 10.1685/2010CAIM487, 1, 1, (2010) 54–71
- [18] Arvind Kumar Gupta and Sapna Sharma, Analysis of the wave properties of a new two-lane continuum model with the coupling effect, Chin. Phys. B Vol. 21, No. 1 (2012) 015201
- [19] Joanne V. Morgan, Numerical Methods For Macroscopic Traffic Models, PhD Thesis 2002, The University of Reading, UK