



## ON THE IITII.C<sub>1</sub> SUMMABILITY OF A SEQUENCE OF FOURIER COEFFICIENTS

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**Abstract:** Mohanty and Nanda (1956) were the first to establish a result for the  $(c,1)$  ie  $c_1$  - summability of the sequence  $\{n B_n(x)\}$ . Varshney (1959) improved the result for  $(N, \frac{1}{n+1})$ :  $c_1$  summability which was generalised by various investigators, such as Sharma (1969), (1970), Dwivedi (1971), Lal (1971), Prasad (1981). In this paper we have discussed  $(a_n, k)$   $c_1$  - summability of the sequence  $\{n, B_n(k)\}$ , which includes the result due to Prasad (1981).

**Keywords:** IITII.C<sub>1</sub> Summability, Fourier Coefficient

### I. INTRODUCTION

Let  $\sum u_n$  be a given infinite series with the sequence of partial sum  $\{s_n\}$ . Let IITII  $\equiv (a_{n,k})$  be infinite triangular with real constants. The sequence to sequence transformation

$$t_n = \sum_{k=0}^n a_{n,k} s_k, n = 0, 1, 2, \dots$$

Defines the T-transform of the sequence  $\{s_n\}$ . Recall that the matrix element  $a_{n,k} = 0$  for each  $k > n$  then the matrix is called triangular matrix. The series  $\sum u_n$  is said to be T- summable to s.

$$\text{if } \lim_{n \rightarrow \infty} t_n = s \text{ as } n \rightarrow \infty$$

The regularity conditions for T - method are :

- 1) There exists a constant  $k : \sum_{k=0}^n |a_{n,k}| < k$  for each  $n$
- 2) For every  $k$ ,  $\lim_{n \rightarrow \infty} a_{n,k} = 0$ , and
- 3)  $\lim_{n \rightarrow \infty} \sum_{k=0}^n a_{n,k} = 1$

The matrix T - reduces to Norlund matrix generated by the sequence of coefficient  $\{p_n\}$  if

$$a_{n,k} = [P_{n-k} \setminus p_n] \text{ if } k \leq n; \quad 0 \text{ if } k > n,$$

$$\text{we have } p_n = \sum_{r=0}^n p_r \neq 0$$

if the method of summability IITII is applied to cesaro means of order one another method of summability IITII.C<sub>1</sub> is obtained.

### II. BASIC TERMINOLOGY

Let  $f(x)$  be a periodic function with period  $2\pi$  and Integrable in the sense of Lebesgue over an interval  $(-\pi, \pi)$ . Let the fourier series of  $f(x)$  be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (2.1)$$

$$\text{And the series conjugate to (2.1) is } \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \quad (2.2)$$

We write

$$\phi(t) = f(x+t) - f(x-t)$$

$$\psi(t) = f(x+t) - (f(x-t) - l);$$

$$\psi_1(t) = \int_0^1 |\psi(u)| du$$

$$A_{n,\tau} = \sum_{k=\tau}^n [a_{n,k}]$$

and  $\tau = [\frac{1}{t}]$  the integral part of  $\frac{1}{t}$  where  $l$  is a constant.

### III. SOME THEOREMS

Mohanty and Nanda (1954) proved the following theorem.

**Theorem A:** If  $\psi(t) = O[\frac{1}{\log(\frac{1}{t})}]$  as  $t \rightarrow 0$ ;  $a_n = O(n^{-b})$ ;  $b_n = O(n^{-\delta})$ ,  $0 < \delta < 1$

Then the sequence  $\{n B_n(x)\}$  is summable  $(c,1)$  to the value  $\frac{l}{\pi}$ .

From this result they have deduced a well known criterion, the Hardy and Little wood test for the convergence of the conjugate series (2.2). Varshney (1959) improved theorem A in the following form.

**Theorem B :** If  $\int_0^t |\psi(u)| du = \psi_1(t) = O\left(\frac{t}{\log(1/t)}\right)$  as  $t \rightarrow 0$

Then the sequence  $\{n B_n(x)\}$  is summable  $(N, \frac{1}{n+1})$  to the value  $\frac{l}{\pi}$ .

Various authors such as Sharma (1969, 1970), Lal (1971), Dwivedi (1971) have generalised the above theorem of varshney in different directions. Recently Prasad (1981) further extended the theorem of varshney to  $(N, p_n)$  summability of the sequence  $\{n B_n(x)\}$ . He proved

**Theorem C:-** Let  $P(u)$  be a monotonic decreasing and positive for  $u \geq 0$ , let  $p_n = p(n)$  and  $p(u) = \int_0^u p(x) dx$ , such that  $p(u) \rightarrow \infty$  as  $u \rightarrow \infty$

Let  $\alpha(t)$  be a positive non-decreasing function of  $t$

If  $\psi_1(t) = \int_0^t |\psi(u)| du = O\left(\frac{t}{\alpha(1/t)}\right)$  as  $t \rightarrow 0$ .

Then a sufficient condition that the sequence  $\{n B_n(x)\}$  be summable  $(N, p_n)C_1$  to the value  $l/\pi$  is that  $\int_1^u \frac{p(t)}{t\alpha(t)} dt = O(p(u))$  as  $u \rightarrow \infty$

4)- Now we extend the above theorem to  $II T II$  summabilities of the  $\{n B_n(x)\}$ .

We prove the following main theorem in next section.

#### IV. MAIN THEOREM

If  $II T II \equiv (a_{n,k})$  be an infinite triangular matrix with  $a_{n,k} \neq 0$  and  $a_{n,k}$  defined by  $a_{n,k} = a_n(K)$ ,  $a_n(u)$  being a strictly positive monotone non increasing function and

$A(n, n-u) = \int_{t=0}^u a_n(n-t) dt \rightarrow 1$  as  $n \rightarrow \infty$  For fixed  $u \geq 0$

Let  $\alpha(t)$  be a positive non-decreasing function of

$t$  if  $\psi_1(t) = O\left(\frac{\alpha(1/t)t}{\log(1/t)}\right)$  as  $t \rightarrow 0$

then the sequence  $\{n B_n(x)\}$  be summable  $II T II$   $C_1$  to

the value  $l/\pi$  such that  $\frac{\alpha(n)}{\log n} \rightarrow 0$  as  $n \rightarrow \infty$

$$= O(1)$$

#### V. SOME LEMMAS

We require the following lemmas to prove the theorem:-

**Lemma 1** (Kishore and Hotta (1971)) - If  $\{a_{n,k}\}_{k=0}^n$  is non negative and non decreasing with respect to  $k$  then for  $0 \leq a \leq b \leq \delta$ ,  $0 \leq t \leq \pi$  and any  $n$ .

$$|\sum_{a=n-k}^b a_n e^{i(n-k)t}| \leq K A_n, n-\tau \quad (5.1)$$

**Lemma 2** - If  $0 \leq t \leq \frac{1}{n}$ , then

$$|Q_n(t)| = \left| \frac{1}{\pi} \sum_{k=1}^n a_{n,k} \left( \frac{\sin kt}{k} - \frac{\cos kt}{t} \right) \right| = O(n) \quad (5.2)$$

**Proof** - for  $0 \leq t \leq \frac{1}{n}$  we have

$$|Q_n(t)| = O\left(\sum_{k=1}^n k^2 t(a_n, k)\right) = O\left(n \sum_{k=1}^n a_{n,k}\right) = O(n)$$

**Lemma 3** - if  $0 < t \leq \pi$   $|Q_n(t)| = O\left(\frac{A_{n,n-\tau}}{t}\right)$

**Proof** - By Abel's transformation, we have

$$\begin{aligned} |Q_n(t)| &= \left| \frac{1}{\pi} \sum_{k=1}^n a_{n,k} \left( \frac{\sin kt}{k} - \frac{\cos kt}{t} \right) \right| \\ &= O\left(\sum_{k=0}^n a_{n,k} \frac{\sin kt}{k} \right) + O\left(\frac{A_{n,n-\tau}}{t}\right) \\ &= O\left(\frac{1}{t} \sum_{k=0}^n a_{n-k} \frac{\sin(n-k)t}{(n-k)t} \right) + O\left(\frac{1}{t}\right) \sum_{k=\tau+1}^{n-1} a_{n-k} \frac{\sin(n-k)t}{(n-k)t} + O\left(\frac{A_{n,n-\tau}}{t}\right) + O\left(\frac{1}{t} \sum_{k=0}^{\tau} a_{n,n-k}\right) \\ &+ O\left(\frac{1}{t^2} \sum_{k=1}^{n-2} |\Delta K a_{n,n-k}|\right) + O(t^2 a_{n,n-\tau-1}) + O(t^2 a_{n,n-k}) + O(t^{-1} A_{n,n-\tau}) \\ &= O\left(\frac{A_{n,n-\tau}}{t}\right) \end{aligned}$$

## VI. PROOF OF MAIN THEOREM

If we denote the  $c_1$  transformation of the sequence  $\{n B_n(x)\}$  by  $t_n$ , we have after Mohanty and Nanda (1954), that

$$\begin{aligned} t_n - l/\pi &= \frac{1}{\pi} \sum_{k=1}^n k B_k(x) - l/\pi \\ &= \frac{1}{\pi} \int_0^\pi \psi(t) \left[ \frac{1}{4n} \frac{\sin nt}{\sin^2 t/2} - \frac{1}{2} \frac{\cos nt}{\tan t/2} \right] dt + \frac{1}{2\pi} \int_0^\pi \psi(t) \sin nt dt + O(1) \end{aligned} \quad (6.1)$$

As the second integral becomes  $o(1)$ , by Riemann lebesgue theorem. Also sine method of summable under consideration is regular. we have to show that under the conditions of the theorem

$$\begin{aligned} I &= \int_0^\pi \psi \left( \frac{t}{\pi} \right) \sum_{k=1}^n a_{n,k} \left( \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right) dt \\ &= O(1) \text{ as } n \rightarrow \infty \end{aligned} \quad (6.2)$$

$$\text{By lemma (2) } Q_n(t) \equiv \frac{1}{\pi} \sum_{k=1}^n a_{n,k} \left( \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right)$$

$$\begin{aligned} \text{Thus } I &= \int_0^\pi \psi(t) \cdot Q_n(t) dt \\ &= \left( \int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi \right) \psi(t) Q_n(t) dt \\ &= I_1 + I_2 + I_3 \quad \text{say where } 0 < \delta < \pi \end{aligned}$$

Using lemma (2) and (4.2) we have

$$\begin{aligned} I_1 &= \int_0^{1/n} \psi(t) Q_n(t) dt = O \left( n \int_0^{1/n} |\psi(t)| dt \right) \\ &= O \left( \frac{\alpha(n)}{\log n} \right) \rightarrow O(1) \text{ as } n \rightarrow \infty \end{aligned}$$

Using lemma 3, (4.2) and (4.3) we have

$$\begin{aligned} I_2 &= \int_{1/n}^\delta \varphi(t) Q_n(t) dt \\ &= O \left( \int_{1/n}^\delta |\psi(t)| \cdot \frac{A_{n,n-\tau}}{t} dt \right) \\ &= O \left( \psi_{1/n}(t) \frac{A_{n,n-\tau}}{t} \right)_{1/n}^\delta + O \left( \int_{1/n}^\delta \psi(t) d \left( \frac{A_{n,n-\tau}}{t} \right) \right) \\ &= O \left( \frac{\alpha(1/t)t}{\log^{1/t}} \frac{A_{n,n-\tau}}{t} \int_{1/n}^\delta + O \left( \int_{1/n}^\delta O \left( \frac{\alpha(1/t)t}{\log^{1/t}} \right) d \left( \frac{\log(1/t) A_{n,n-\tau}}{t \log^{1/t}} \right) \right) \right) \\ &= O \left( \frac{\alpha(n)}{\log n} \right) + O(1) \left( \int_{1/n}^\delta \frac{\alpha(1/t)t}{\log^{1/t}} \log \left( \frac{1}{t} \right) d \left( \frac{A_{n,n-\tau}}{t \log^{1/t}} \right) + O(1) \int_{1/n}^\delta \frac{\alpha(1/t)t}{\log^{1/t}} : \frac{A_{n,n-\tau}}{t \log^{1/t}} d \log \left( \frac{1}{t} \right) \right) \\ &= O \left\{ \frac{\alpha(n)}{\log n} \right\} + O(1) \int_{1/n}^\delta \frac{\alpha(1/t)t}{\log^{1/t}} d \left( \frac{\log(1/t)}{\log(1/t)} \right) + O(1) \int_{1/n}^\delta \frac{A_{n,n-\tau}}{t \log^{1/t}} dt \\ &= O \left( \frac{\alpha(N)}{\log n} \right) + O(1) \left( \frac{t \alpha(1/t)}{t} \frac{A_{n,n-\tau}}{\log(1/t)} \right)_{1/n}^\delta + O(1) \int_{1/n}^\delta \frac{A_{n,n-\tau}}{t \log^{1/t}} dt \\ &= O(1) + O(1) + O(1) \\ &= O(1) \quad \text{as } \left( \frac{t \alpha(1/t)}{\log^{1/t}} \right)_{1/n}^\delta \rightarrow O(1) \text{ And } \frac{A_{n,n-\tau}}{t} \rightarrow O(1) \text{ as } n \rightarrow \infty \end{aligned}$$

Since the method of summation is regular we have by Riemann – Lebesgue theorem

$$I_3 = \int_\delta^\pi \psi(t) \cdot Q_n(t) dt = O(1) \text{ as } h \rightarrow \infty$$

This completes the proof of main theorem.

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