



## Fixed Point Theorem in Complex Valued Metric Spaces for Continuity and Compatibility

<sup>1</sup>Mala Hakwadiya, <sup>2</sup>R K Gujtiya, <sup>3</sup>Dheeraj Kumari Mali

<sup>2</sup> Associate Professor & Head, Department of Mathematics,  
Govt. P. G. College, Neemuch, INDIA

<sup>1,3</sup> Research Scholar, Pacific Academy Of Higher Education  
and Research University Udaipur, Rajasthan, INDIA

**Abstract:** In this paper, we prove some common fixed point theorems in complex valued metric spaces for weakly compatibility and continuity. Which generalized the well known result.

**Keywords:** Common fixed point, complex valued metric space, Weakly compatible mapping.

**Mathematics Subject Classification:** 47H10, 54H25.

### I. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric space, which is a generalized of the classical metric space and established some fixed point result for mappings satisfying a rational inequality. In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion. Let  $\mathbb{C}$  be the set of complex numbers and  $z_1, z_2 \in \mathbb{C}$ : Define a partial order  $\preceq$  on  $\mathbb{C}$  as follows:  $z_1 \preceq z_2$  if and only if  $\text{Re}(z_1) \leq \text{Re}(z_2)$ ;  $\text{Im}(z_1) \leq \text{Im}(z_2)$ . Consequently, one can infer that  $z_1 \preceq z_2$  if one of the following conditions is satisfied: (i)  $\text{Re}(z_1) = \text{Re}(z_2)$ ,  $\text{Im}(z_1) < \text{Im}(z_2)$  (ii)  $\text{Re}(z_1) < \text{Re}(z_2)$ ,  $\text{Im}(z_1) = \text{Im}(z_2)$  (iii)  $\text{Re}(z_1) < \text{Re}(z_2)$ ,  $\text{Im}(z_1) < \text{Im}(z_2)$  (iv)  $\text{Re}(z_1) = \text{Re}(z_2)$ ,  $\text{Im}(z_1) = \text{Im}(z_2)$ . In particular, we write  $z_1 \prec z_2$  if  $z_1 \neq z_2$  and one of (i), (ii), and (iii) is satisfied and we write  $z_1 \preceq z_2$  if only (iii) is satisfied. Notice that  $0 \preceq z_1 \prec z_2 \Rightarrow |z_1| < |z_2|$  and  $z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$ . Recently, Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces.

### II. PRELIMINARIES

**Definition 2.1.** [4] Let  $X$  be a nonempty set whereas  $\mathbb{C}$  be the set of complex numbers. Suppose that the mapping  $d: X \times X \rightarrow \mathbb{C}$  satisfies following conditions:

- (1)  $0 \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (3)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a complex valued metric on  $X$ , and  $(X, d)$  is called a complex valued metric space.

**Definition 2.2.** [8] Let  $(X, d)$  be a complex valued metric space and  $\{x_n\}$  be a sequence in  $X$ .

- (1) If for every  $c \in \mathbb{C}$  with  $0 < c$ , there exists  $N \in \mathbb{N}$  such that  $d(x_n, x) \prec c$  for all  $n \geq N$  then  $\{x_n\}$  is said to be convergent to  $x \in X$ , and we denote this by  $x_n \rightarrow x$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} x_n = x$ .
- (2) If for every  $c \in \mathbb{C}$  with  $0 < c$ , there exists  $N \in \mathbb{N}$  such that  $d(x_n, x_{n+m}) \prec c$  for all  $n \geq N$ , where  $m \in \mathbb{N}$ , then  $\{x_n\}$  is said to be Cauchy sequence.
- (3) If every Cauchy sequence in  $X$  is convergent, then  $(X, d)$  is said to be a complete complex valued metric space.

**Lemma 2.3.** [8] Let  $(X, d)$  be a complex valued metric space and  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  converges to  $x$  if and only if  $|d(x_n, x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.4.** [8] Let  $(X, d)$  be a complex valued metric space and  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is a Cauchy sequence if and only if  $|d(x_n, x_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ , where  $m \in \mathbb{N}$ .

**Definition 2.5.** [8] Let  $f$  and  $g$  be two self-mappings of a metric space  $(X, d)$ . Then a pair  $(f, g)$  is said to be weakly compatible if they commute at coincidence points.

**Definition 2.6:** [3] Let  $A$  and  $S$  be mappings from a complete metric space  $X$  into itself. The mappings  $A$  and  $S$  are said to be compatible if  $\lim_{n \rightarrow \infty} d(ASx_n, SAx_n) = 0$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$  for some  $t \in X$ .

### III. Main Result

**Theorem 3.1 :** Let  $(X, d)$  be a complex valued metric space and  $A, B, D, M, S$  and  $T$  be six self mappings in  $X$  satisfying the condition:

1.  $S(X) \subset BD(X)$  and  $T(X) \subset AM(X)$
2. For each  $x, y \in X$ , such that  $x \neq y$ ,  $d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx) \neq 0$ , where  $\alpha, \beta, \gamma$  and  $\eta$  are non negative real number with  $\alpha + \beta + 2\gamma + \eta < 1$ , or  $d(Sx, Ty) = 0$  if  $d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx) = 0$ , such that

$$d(Sx, Ty) \lesssim \alpha \left[ \frac{d(AMx, Sx) + d(BDy, Ty)d(AMx, Sx)}{1 + d(Sx, Ty)} \right] + \beta \max\{d(AMx, BDy), d(AMx, Sx), d(BDy, Sx)\} + \gamma \{d(BDy, Ty) + d(Ty, AMx) + d(Sx, BDy)\} + \eta \left[ \frac{d(Ty, BDy)d(Sx, AMx)}{d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx)} \right]$$

3. The pair  $(AM, S)$  and  $(BD, T)$  are commute.

4. The pair  $(AM, S)$  and  $(BD, T)$  are weakly compatible.

Then  $A, B, D, M, S$  and  $T$  have a unique common fixed point.

**Proof :** Let  $x_0 \in X$ . Since  $S(X) \subset BD(X)$  and  $T(X) \subset AM(X)$ , define for each  $n \geq 0$ , the sequence  $\{y_n\}$  in  $X$  by

$$y_{2n+1} = Sx_{2n} = BDx_{2n+1} \text{ and } y_{2n+2} = Tx_{2n+1} = AMx_{2n+2} \quad n = 0, 1, 2, \dots$$

$$\begin{aligned} \text{Then } d(y_{2n+1}, y_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \lesssim \alpha \left[ \frac{d(AMx_{2n}, Sx_{2n}) + d(BDx_{2n+1}, Tx_{2n+1})d(AMx_{2n}, Sx_{2n})}{1 + d(Sx_{2n}, Tx_{2n+1})} \right] \\ &\quad + \beta \max\{d(AMx_{2n}, BDx_{2n+1}), d(AMx_{2n}, Sx_{2n}), d(BDx_{2n+1}, Sx_{2n})\} \\ &\quad + \gamma \{d(BDx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n}) + d(Sx_{2n}, BDx_{2n+1})\} \\ &\quad + \eta \left[ \frac{d(Tx_{2n+1}, BDx_{2n+1})d(Sx_{2n}, AMx_{2n})}{d(Tx_{2n+1}, AMx_{2n}) + d(Sx_{2n}, BDx_{2n+1}) + d(BDx_{2n+1}, AMx_{2n})} \right] \\ d(y_{2n+1}, y_{2n+2}) &\lesssim \alpha \left[ \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n+2})} \right] \\ &\quad + \beta \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}), d(y_{2n+1}, y_{2n+1})\} \\ &\quad + \gamma \{d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})\} \\ &\quad + \eta \left[ \frac{d(y_{2n+2}, y_{2n+1})d(y_{2n+1}, y_{2n})}{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n})} \right] \\ d(y_{2n+1}, y_{2n+2}) &\lesssim \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1}) + \eta d(y_{2n}, y_{2n+1}) \\ d(y_{2n+1}, y_{2n+2}) &\lesssim (\alpha + \beta + \gamma + \eta) d(y_{2n}, y_{2n+1}) \end{aligned} \quad (3.1)$$

That is  $|d(y_{2n+1}, y_{2n+2})| \leq (\alpha + \beta + \gamma + \eta) |d(y_{2n}, y_{2n+1})|$ , Similarly

$$\begin{aligned} d(y_{2n+3}, y_{2n+4}) &= d(Sx_{2n+2}, Tx_{2n+3}) \lesssim \alpha \left[ \frac{d(AMx_{2n+2}, Sx_{2n+2}) + d(BDx_{2n+3}, Tx_{2n+3})d(AMx_{2n+2}, Sx_{2n+2})}{1 + d(Sx_{2n+2}, Tx_{2n+3})} \right] \\ &\quad + \beta \max\{d(AMx_{2n+2}, BDx_{2n+3}), d(AMx_{2n+2}, Sx_{2n+2}), d(BDx_{2n+3}, Sx_{2n+2})\} \\ &\quad + \gamma \{d(BDx_{2n+3}, Tx_{2n+3}) + d(Tx_{2n+3}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+3})\} \\ &\quad + \eta \left[ \frac{d(Tx_{2n+3}, BDx_{2n+3})d(Sx_{2n+2}, AMx_{2n+2})}{d(Tx_{2n+3}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+3}) + d(BDx_{2n+3}, AMx_{2n+2})} \right] \\ d(y_{2n+3}, y_{2n+4}) &\lesssim \alpha \left[ \frac{d(y_{2n+2}, y_{2n+3}) + d(y_{2n+3}, y_{2n+4})d(y_{2n+2}, y_{2n+3})}{1 + d(y_{2n+3}, y_{2n+4})} \right] \\ &\quad + \beta \max\{d(y_{2n+2}, y_{2n+3}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+3}, y_{2n+3})\} \\ &\quad + \gamma \{d(y_{2n+3}, y_{2n+4}) + d(y_{2n+4}, y_{2n+2}) + d(y_{2n+3}, y_{2n+3})\} \\ &\quad + \eta \left[ \frac{d(y_{2n+4}, y_{2n+3})d(y_{2n+3}, y_{2n+2})}{d(y_{2n+4}, y_{2n+2}) + d(y_{2n+3}, y_{2n+3}) + d(y_{2n+3}, y_{2n+2})} \right] \\ d(y_{2n+3}, y_{2n+4}) &\lesssim \alpha d(y_{2n+2}, y_{2n+3}) + \beta d(y_{2n+2}, y_{2n+3}) + \gamma d(y_{2n+2}, y_{2n+3}) + \eta d(y_{2n+2}, y_{2n+3}) \\ d(y_{2n+3}, y_{2n+4}) &\lesssim (\alpha + \beta + \gamma + \eta) d(y_{2n+2}, y_{2n+3}) \end{aligned} \quad (3.2)$$

That is  $|d(y_{2n+3}, y_{2n+4})| \leq (\alpha + \beta + \gamma + \eta) |d(y_{2n+2}, y_{2n+3})|$

Therefore from (3.1) and (3.2)  $d(y_n, y_{n+1}) \lesssim (\alpha + \beta + \gamma + \eta) d(y_{n+1}, y_n)$  If  $\delta = \alpha + \beta + \gamma + \eta < 1$ . Then it

can be concluded that  $d(y_n, y_{n+1}) \lesssim \delta d(y_{n-1}, y_n)$

$$d(y_n, y_{n+1}) \lesssim \delta^2 d(y_{n-2}, y_{n-1}) \lesssim \dots \lesssim \delta^n d(y_0, y_1)$$

Now for all  $m > n$ , we have  $d(y_m, y_n) \lesssim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{m-1}, y_m)$

$$d(y_m, y_n) \lesssim \delta^n d(y_0, y_1) + \delta^{n+1} d(y_0, y_1) + \dots + \delta^{m-1} d(y_0, y_1)$$

Therefore we have,  $|d(y_m, y_n)| \leq \frac{\delta^n}{1 - \delta} |d(y_0, y_1)|$

Hence  $|d(y_m, y_n)| \leq \frac{\delta^n}{1 - \delta} |d(y_0, y_1)| \rightarrow 0$  as  $m, n \rightarrow \infty$ . that is  $\lim_{m, n \rightarrow \infty} |d(y_m, y_n)| = 0$ .

Hence  $\{y_n\}$  is a Cauchy sequence. Since  $X$  is completed, so  $\{y_n\}$  converges to some point  $z$ . thus subsequence  $\{Sx_{2n}\}, \{BDx_{2n+1}\}, \{Tx_{2n+1}\}$  and  $\{AMx_{2n+2}\}$  also converges to  $z$ . that is  $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} BDx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} AMx_{2n+2} = z$ . There exists some  $u \in X$  such that  $y_n \rightarrow u$  as  $n \rightarrow \infty$ .  $Su = AMu = BDu = Tu = z$ . Since The pair  $(AM, S)$  and  $(BD, T)$  are weakly compatible. Then they commute at their coincidence point. Hence  $Sz = S(AMu) = AM(Su) = AMz$ . And  $BDz = BD(Tu) = T(BDu) = Tz$ .

**Case (I) :** Now, we shall show that  $Tz = Sz$ . form (2) putting  $x = z$  and  $y = x_{2n+1}$  we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) &\lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BDx_{2n+1}, Tx_{2n+1})d(AMz, Sz)}{1 + d(Sz, Tx_{2n+1})} \right] \\ &\quad + \beta \max\{d(AMz, BDx_{2n+1}), d(AMz, Sz), d(BDx_{2n+1}, Sz)\} \\ &\quad + \gamma \{d(BDx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, AMz) + d(Sz, BDx_{2n+1})\} \\ &\quad + \eta \left[ \frac{d(Tx_{2n+1}, BDx_{2n+1})d(Sz, AMz)}{d(Tx_{2n+1}, AMz) + d(Sz, BDx_{2n+1}) + d(BDx_{2n+1}, AMz)} \right] \\ d(Sz, y_{2n+2}) &\lesssim \alpha \left[ \frac{d(Sz, Sz) + d(y_{2n+1}, y_{2n+2})d(Sz, Sz)}{1 + d(Sz, y_{2n+2})} \right] + \beta \max\{d(Sz, y_{2n+1}), d(Sz, Sz), d(y_{2n+1}, Sz)\} \\ &\quad + \gamma \{d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, Sz) + d(Sz, y_{2n+1})\} + \eta \left[ \frac{d(y_{2n+2}, y_{2n+1})d(Sz, Sz)}{d(y_{2n+2}, Sz) + d(Sz, y_{2n+1}) + d(y_{2n+1}, Sz)} \right] \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} d(Sz, z) &\lesssim \alpha \left[ \frac{d(Sz, Sz) + d(z, z)d(Sz, Sz)}{1 + d(Sz, z)} \right] + \beta \max\{d(Sz, z), d(Sz, Sz), d(z, Sz)\} \\ &\quad + \gamma \{d(z, z) + d(z, Sz) + d(Sz, z)\} + \eta \left[ \frac{d(z, z)d(Sz, Sz)}{d(z, Sz) + d(Sz, z) + d(z, Sz)} \right] \end{aligned}$$

$d(Sz, z) \lesssim \beta d(Sz, z) + 2\gamma d(Sz, z)$ . Then  $d(Sz, z) \lesssim (\beta + 2\gamma)d(Sz, z)$  That is,  $|d(Sz, z)| \leq (\beta + 2\gamma)|d(Sz, z)|$  Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $Sz = z$ . since  $AMz = Sz$  which implies  $AMz = z$ . Now we prove that  $Tz = z$ . from (2), putting  $x = y = z$ , we get

$$\begin{aligned} d(Sz, Tz) &\lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BDz, Tz)d(AMz, Sz)}{1 + d(Sz, Tz)} \right] + \beta \max\{d(AMz, BDz), d(AMz, Sz), d(BDz, Sz)\} \\ &\quad + \gamma \{d(BDz, Tz) + d(Tz, AMz) + d(Sz, BDz)\} + \eta \left[ \frac{d(Tz, BDz)d(Sz, AMz)}{d(Tz, AMz) + d(Sz, BDz) + d(BDz, AMz)} \right] \\ d(z, Tz) &\lesssim \alpha \left[ \frac{d(z, z) + d(Tz, Tz)d(z, z)}{1 + d(z, Tz)} \right] + \beta \max\{d(z, Tz), d(z, z), d(Tz, z)\} \\ &\quad + \gamma \{d(Tz, Tz) + d(Tz, z) + d(z, Tz)\} + \eta \left[ \frac{d(Tz, Tz)d(z, z)}{d(Tz, z) + d(z, Tz) + d(Tz, z)} \right] \end{aligned}$$

$d(z, Tz) \lesssim \beta d(z, Tz) + 2\gamma d(z, Tz)$ . Then  $d(Tz, z) \lesssim (\beta + 2\gamma)d(Tz, z)$  That is,  $|d(Tz, z)| \leq (\beta + 2\gamma)|d(Tz, z)|$  Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $Tz = z$ . since  $BDz = Tz$  which implies  $BDz = z$ . Now we prove that  $Mz = z$ . from (2), putting  $x = Mz$  and  $y = z$ , we get

$$\begin{aligned} d(S(Mz), Tz) &\lesssim \alpha \left[ \frac{d(AM(Mz), S(Mz)) + d(BDz, Tz)d(AM(Mz), S(Mz))}{1 + d(S(Mz), Tz)} \right] \\ &\quad + \beta \max\{d(AM(Mz), BDz), d(AM(Mz), S(Mz)), d(BDz, S(Mz))\} \\ &\quad + \gamma \{d(BDz, Tz) + d(Tz, AM(Mz)) + d(S(Mz), BDz)\} \\ &\quad + \eta \left[ \frac{d(Tz, BDz)d(S(Mz), AM(Mz))}{d(Tz, AM(Mz)) + d(S(Mz), BDz) + d(BDz, AM(Mz))} \right] \\ d(Mz, z) &\lesssim \alpha \left[ \frac{d(Mz, Mz) + d(z, z)d(Mz, Mz)}{1 + d(Mz, z)} \right] + \beta \max\{d(Mz, z), d(Mz, Mz), d(z, Mz)\} \\ &\quad + \gamma \{d(z, z) + d(z, Mz) + d(Mz, z)\} + \eta \left[ \frac{d(z, z)d(Mz, Mz)}{d(z, Mz) + d(Mz, z) + d(z, Mz)} \right] \end{aligned}$$

$d(Mz, z) \lesssim \beta d(Mz, z) + 2\gamma d(Mz, z)$ . That is,  $|d(Mz, z)| \leq (\beta + 2\gamma)|d(Mz, z)|$  Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $Mz = z$ . since  $AMz = z$  which implies  $Az = z$ . Now we prove that  $Dz = z$ . from (2), putting  $x = z$  and  $y = Dz$ , we get

$$\begin{aligned} d(Sz, T(Dz)) &\lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BD(Dz), T(Dz))d(AMz, Sz)}{1 + d(Sz, T(Dz))} \right] \\ &\quad + \beta \max\{d(AMz, BD(Dz)), d(AMz, Sz), d(BD(Dz), Sz)\} \\ &\quad + \gamma \{d(BD(Dz), T(Dz)) + d(T(Dz), AMz) + d(Sz, BD(Dz))\} \\ &\quad + \eta \left[ \frac{d(T(Dz), BD(Dz))d(Sz, AMz)}{d(T(Dz), AMz) + d(Sz, BD(Dz)) + d(BD(Dz), AMz)} \right] \\ d(z, Dz) &\lesssim \alpha \left[ \frac{d(z, z) + d(Dz, Dz)d(z, z)}{1 + d(z, Dz)} \right] + \beta \max\{d(z, Dz), d(z, z), d(Dz, z)\} \\ &\quad + \gamma \{d(Dz, Dz) + d(Dz, z) + d(z, Dz)\} + \eta \left[ \frac{d(Dz, Dz)d(z, z)}{d(Dz, z) + d(z, Dz) + d(Dz, z)} \right] \end{aligned}$$

$d(z, Dz) \lesssim \beta d(z, Dz) + 2\gamma d(z, Dz)$ . Then  $d(z, Dz) \lesssim (\beta + 2\gamma)d(z, Dz)$  That is,  $|d(z, Dz)| \leq (\beta + 2\gamma)|d(z, Dz)|$  Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $Dz = z$ . since  $BDz = z$  which implies  $Bz = z$ . Therefore  $z$  is a common fixed point of  $A, B, D, M, S$  and  $T$ .

**Uniqueness:** Let  $u$  be an another common fixed point of  $A, B, D, M, S$  and  $T$ . Then, we have

$$d(Sz, Tu) \lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BDu, Tu)d(AMz, Sz)}{1 + d(Sz, Tu)} \right] + \beta \max\{d(AMz, BDu), d(AMz, Sz), d(BDu, Sz)\} \\ + \gamma \{d(BDu, Tu) + d(Tu, AMz) + d(Sz, BDu)\} + \eta \left[ \frac{d(Tu, BDu)d(Sz, AMz)}{d(Tu, AMz) + d(Sz, BDu) + d(BDu, AMz)} \right] \\ d(z, u) \lesssim \alpha \left[ \frac{d(z, z) + d(u, u)d(z, z)}{1 + d(z, u)} \right] + \beta \max\{d(z, u), d(z, z), d(u, z)\} \\ + \gamma \{d(u, u) + d(u, z) + d(z, u)\} + \eta \left[ \frac{d(u, u)d(z, z)}{d(u, z) + d(z, u) + d(u, z)} \right]$$

$d(z, u) \lesssim \beta d(z, u) + 2\gamma d(z, u)$ . Then  $d(z, u) \lesssim (\beta + 2\gamma)d(z, u)$  That is,  $|d(z, u)| \leq (\beta + 2\gamma)|d(z, u)|$  Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $z = u$ . Hence  $z$  is a unique common fixed point of  $A, B, D, M, S$  and  $T$ .

**Case II:** we consider the case :  $d(Tx_{2n+1}, AMx_{2n}) + d(Sx_{2n}, BDx_{2n+1}) + d(BDx_{2n+1}, AMx_{2n}) = 0$  (for any  $n$ ) implies that  $d(Sx_{2n}, Tx_{2n+1}) = 0$ . So that  $y_{2n} = Sx_{2n} = y_{2n+1} = BDx_{2n+1} = Tx_{2n+1} = AMx_{2n+2} = y_{2n+2}$ . thus we have  $y_{2n+1} = Sx_{2n} = AMx_{2n} = y_{2n}$  there exists  $n_1$  and  $m_1$  such that  $n_1 = Sm_1 = AMm_1 = m_1$ . Similarly  $y_{2n+2} = Tx_{2n+1} = BDx_{2n+1} = y_{2n+1}$ . there exists  $n_2$  and  $m_2$  such that  $n_2 = Tm_2 = BDM_2 = m_2$ . As  $d(Tm_2, AMm_1) + d(Sm_1, BDM_2) + d(BDM_2, AMm_1) = 0$  implies that  $d(Sm_1, Tm_2) = 0$ , so that  $n_1 = Sm_1 = AMm_1 = Tm_2 = BDM_2 = n_2$ . Which in turn yields that  $n_1 = Sn_1 = AMm_1 = AMn_1$  similarly, one can also have  $n_2 = Tn_2 = BDM_2$ . As  $n_1 = n_2$ , implies  $n_1 = Sn_1 = Tn_1 = BDM_1$ , therefore  $n_1 = Sn_1 = An_1 = Mn_1 = Tn_1 = Bn_1 = Dn_1$ . Hence  $n_1 = n_2$ , is common fixed point.

**Uniqueness :** Let  $v_1$  is an another common fixed point of  $A, B, D, M, S$  and  $T$ . Then we have  $v_1 = Sv_1 = Av_1 = Mv_1 = Tv_1 = Bv_1 = Dv_1$ . Therefore  $d(Tv_1, AMn_1) + d(Sn_1, BDv_1) + d(BDv_1, AMn_1) = 0$  so that  $d(n_1, v_1) = 0$ . Hence  $d(Sn_1, Tv_1) = 0$  this implies that  $n_1 = v_1$ . Hence  $n_1$  is a unique common fixed point of  $A, B, D, M, S$  and  $T$ .

**Corollary:** Let  $(X, d)$  be a complex valued metric space and  $D, M, S$  and  $T$  be six self mappings in  $X$  satisfying the condition:

1.  $S(X) \subset D(X)$  and  $T(X) \subset M(X)$

2. For each  $x, y \in X$ , such that  $x \neq y$ ,  $d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx) \neq 0$ , where  $\alpha, \beta, \gamma$  and  $\eta$  are non negative real number with  $\alpha + \beta + 2\gamma + \eta < 1$ , or  $d(Sx, Ty) = 0$  if  $d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx) = 0$ , such that

$$d(Sx, Ty) \lesssim \alpha \left[ \frac{d(Mx, Sx) + d(Dy, Ty)d(Mx, Sx)}{1 + d(Sx, Ty)} \right] + \beta \max\{d(Mx, Dy), d(Mx, Sx), d(Dy, Sx)\} \\ + \gamma \{d(Dy, Ty) + d(Ty, Mx) + d(Sx, Dy)\} + \eta \left[ \frac{d(Ty, Dy)d(Sx, Mx)}{d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx)} \right]$$

3. The pair  $(M, S)$  and  $(D, T)$  are weakly compatible.

Then  $D, M, S$  and  $T$  have a unique common fixed point.

**Corollary :** If  $(M, S)$  and  $(D, T)$  are four commuting self mappings defined on a complete complex valued metric space  $(X, d)$  satisfying the condition:

$$d(S^m x, T^n y) \lesssim \alpha \left[ \frac{d(M^m x, S^m x) + d(D^n y, T^n y)d(M^m x, S^m x)}{1 + d(S^m x, T^n y)} \right] \\ + \beta \max\{d(M^m x, D^n y), d(M^m x, S^m x), d(D^n y, S^m x)\} + \gamma \{d(D^n y, T^n y) + d(T^n y, M^m x) + d(S^m x, D^n y)\} \\ + \eta \left[ \frac{d(T^n y, D^n y)d(S^m x, M^m x)}{d(T^n y, M^m x) + d(S^m x, D^n y) + d(D^n y, M^m x)} \right]$$

For each  $x, y \in X$ , such that  $x \neq y$ , where  $\alpha, \beta, \gamma$  and  $\eta$  are non negative real number with  $\alpha + \beta + 2\gamma + \eta < 1$ , or  $d(S^m x, T^n y) = 0$  if  $d(T^n y, M^m x) + d(S^m x, D^n y) + d(D^n y, M^m x) = 0$ , Then  $D, M, S$  and  $T$  have a unique common fixed point.

**Theorem 3.2.** Let  $A, B, D, M, S$  and  $T$  be self mappings of a complete complex valued metric space  $(X, d)$  satisfying conditions

1.  $S(X) \subset BD(X)$  and  $T(X) \subset AM(X)$

2. For each  $x, y \in X$ , where  $\alpha, \beta, \gamma$  and  $\eta$  are non negative real number with  $\alpha + \beta + 2\gamma + \eta < 1$ , such that

$$d(Sx, Ty) \lesssim \alpha \left[ \frac{d(AMx, Sx) + d(BDy, Ty)d(AMx, Sx)}{1 + d(Sx, Ty)} \right] + \beta \max\{d(AMx, BDy), d(AMx, Sx), d(BDy, Sx)\} \\ + \gamma \{d(BDy, Ty) + d(Ty, AMx) + d(Sx, BDy)\} + \eta \left[ \frac{d(Ty, BDy)d(Sx, AMx)}{d(Ty, AMx) + d(Sx, BDy) + d(BDy, AMx)} \right]$$

3. (AM, S) are compatible, and AM or S is continuous and (BD, T) are weakly compatible.
  4. (BD, T) are compatible, and BD or T is continuous and (AM, S) are weakly compatible.
- Then A, B, D, M, S and T have a unique common fixed point.

**Proof :** By above theorem  $\{y_n\}$  is a Cauchy sequence. Since X is completed, so  $\{y_n\}$  is converges to some point z. thus subsequence  $\{Sx_{2n}\}, \{BDx_{2n+1}\}, \{Tx_{2n+1}\}$  and  $\{AMx_{2n+2}\}$  also converges to z. that is

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} BDx_{2n+1} = \lim_{n \rightarrow \infty} AMx_{2n+2} = \lim_{n \rightarrow \infty} Tx_{2n+1} = z \quad (3.3)$$

Assume that S is continuous. Since (AM, S) are compatible, we have

$$\lim_{n \rightarrow \infty} AM(Sx_{2n+2}) = \lim_{n \rightarrow \infty} S(AMx_{2n+2}) = Sz \quad (3.4)$$

Now putting  $x = x_{2n+2}$ ,  $y = x_{2n+1}$  then we have

$$\begin{aligned} d(AM(Sx_{2n+2}), Tx_{2n+1}) \lesssim & \alpha \left[ \frac{d(AMx_{2n+2}, Sx_{2n+2}) + d(BDx_{2n+1}, Tx_{2n+1})d(AMx_{2n+2}, Sx_{2n+2})}{1 + d(Sx_{2n+2}, Tx_{2n+1})} \right] \\ & + \beta \max\{d(AMx_{2n+2}, BDx_{2n+1}), d(AMx_{2n+2}, Sx_{2n+2}), d(BDx_{2n+1}, Sx_{2n+2})\} \\ & + \gamma \{d(BDx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+1})\} \\ & + \eta \left[ \frac{d(Tx_{2n+1}, BDx_{2n+1})d(Sx_{2n+2}, AMx_{2n+2})}{d(Tx_{2n+1}, AMx_{2n+2}) + d(Sx_{2n+2}, BDx_{2n+1}) + d(BDx_{2n+1}, AMx_{2n+2})} \right] \end{aligned}$$

Letting  $n \rightarrow \infty$ , in the above inequality and using (3.3) and (3.4), we get

$$\begin{aligned} d(Sz, z) \lesssim & \alpha \left[ \frac{d(z, z) + d(z, z)d(z, z)}{1 + d(z, z)} \right] + \beta \max\{d(z, z), d(z, z), d(z, z)\} \\ & + \gamma \{d(z, z) + d(z, z) + d(z, z)\} + \eta \left[ \frac{d(z, z)d(z, z)}{d(z, z) + d(z, z) + d(z, z)} \right] \end{aligned}$$

$d(Sz, z) \lesssim 0$  that is  $|d(Sz, z)| \leq 0$  hence  $Sz = z$ . Now putting  $x = z$  and  $y = x_{2n+1}$  in (2) we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) \lesssim & \alpha \left[ \frac{d(AMz, Sz) + d(BDx_{2n+1}, Tx_{2n+1})d(AMz, Sz)}{1 + d(Sz, Tx_{2n+1})} \right] \\ & + \beta \max\{d(AMz, BDx_{2n+1}), d(AMz, Sz), d(BDx_{2n+1}, Sz)\} \\ & + \gamma \{d(BDx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, AMz) + d(Sz, BDx_{2n+1})\} \\ & + \eta \left[ \frac{d(Tx_{2n+1}, BDx_{2n+1})d(Sz, AMz)}{d(Tx_{2n+1}, AMz) + d(Sz, BDx_{2n+1}) + d(BDx_{2n+1}, AMz)} \right] \end{aligned}$$

Letting  $n \rightarrow \infty$ , we have

$$\begin{aligned} d(z, z) \lesssim & \alpha \left[ \frac{d(AMz, z) + d(z, z)d(AMz, z)}{1 + d(z, z)} \right] + \beta \max\{d(AMz, z), d(AMz, z), d(z, z)\} \\ & + \gamma \{d(z, z) + d(z, AMz) + d(z, z)\} + \eta \left[ \frac{d(z, z)d(z, AMz)}{d(z, AMz) + d(z, z) + d(z, AMz)} \right] \end{aligned}$$

$$d(z, z) \lesssim \alpha d(AMz, z) + \beta d(AMz, z) + \gamma d(AMz, z)$$

$d(z, z) \lesssim (\alpha + \beta + \gamma)d(AMz, z)$  That is,  $0 \leq (\alpha + \beta + \gamma)|d(AMz, z)|$ . Then  $|d(AMz, z)| \geq 0$ . hence  $AMz = z$ . Since  $S(X) \subset BD(X)$ , there exists a point  $w \in X$  such that  $Sz = BDw$ . Suppose that  $BDw \neq Tw$ . Now to prove  $BDw = Tw$  and given that  $Sz = z = BDw$ . From (2) putting  $x = z$  and  $y = w$ , we obtain

$$\begin{aligned} d(Sz, Tw) \lesssim & \alpha \left[ \frac{d(AMz, Sz) + d(BDw, Tw)d(AMz, Sz)}{1 + d(Sz, Tw)} \right] + \beta \max\{d(AMz, BDw), d(AMz, Sz), d(BDw, Sw)\} \\ & + \gamma \{d(BDw, Tw) + d(Tw, AMz) + d(Sz, BDw)\} + \eta \left[ \frac{d(Tw, BDw)d(Sz, AMz)}{d(Tw, AMz) + d(Sz, BDw) + d(BDw, AMz)} \right] \end{aligned}$$

$$\begin{aligned} d(z, Tw) \lesssim & \alpha \left[ \frac{d(z, z) + d(z, Tw)d(z, z)}{1 + d(z, Tw)} \right] + \beta \max\{d(z, z), d(z, z), d(z, z)\} \\ & + \gamma \{d(z, Tw) + d(Tw, z) + d(z, z)\} + \eta \left[ \frac{d(Tw, z)d(z, z)}{d(Tw, z) + d(z, z) + d(z, z)} \right] \end{aligned}$$

$$d(z, Tw) \lesssim 2\gamma d(z, Tw) \quad \text{That is, } |d(z, Tw)| \leq 2\gamma |d(z, Tw)|$$

Which is contradiction to  $2\gamma < 1$ . Therefore  $Tw = z$ . hence  $BDw = z = Tw$ . Thus  $BDw = Tw$ . Since BD and T are weakly compatible then  $BDz = BD(Tw) = T(BDw) = Tz$ . Thus z is a coincidence point of BD and T. now to porve  $Tz = z$ , from (2) putting  $x = z$  and  $y = z$ .

$$\begin{aligned} d(Sz, Tz) \lesssim & \alpha \left[ \frac{d(AMz, Sz) + d(BDz, Tz)d(AMz, Sz)}{1 + d(Sz, Tz)} \right] + \beta \max\{d(AMz, BDz), d(AMz, Sz), d(BDz, Sz)\} \\ & + \gamma \{d(BDz, Tz) + d(Tz, AMz) + d(Sz, BDz)\} + \eta \left[ \frac{d(Tz, BDz)d(Sz, AMz)}{d(Tz, AMz) + d(Sz, BDz) + d(BDz, AMz)} \right] \end{aligned}$$

$$d(z, Tz) \lesssim \alpha \left[ \frac{d(z, z) + d(Tz, Tz)d(z, z)}{1 + d(z, Tz)} \right] + \beta \max\{d(z, Tz), d(z, z), d(Tz, z)\} \\ + \gamma \{d(Tz, Tz) + d(Tz, z) + d(z, Tz)\} + \eta \left[ \frac{d(Tz, Tz)d(z, z)}{d(Tz, z) + d(z, Tz) + d(Tz, z)} \right]$$

$$d(z, Tz) \lesssim \beta d(z, Tz) + 2\gamma d(z, Tz)$$

$$d(Tz, z) \lesssim (\beta + 2\gamma)d(Tz, z) \quad \text{That is, } |d(Tz, z)| \leq (\beta + 2\gamma)|d(Tz, z)| \text{ Which is contradiction } \beta + 2\gamma < 1.$$

Therefore  $Tz = z$ . since  $BDz = Tz$  which implies  $BDz = z$ . Now we prove that  $Mz = z$ . from (2), putting  $x = Mz$  and  $y = z$ , we get

$$d(S(Mz), Tz) \lesssim \alpha \left[ \frac{d(AM(Mz), S(Mz)) + d(BDz, Tz)d(AM(Mz), S(Mz))}{1 + d(S(Mz), Tz)} \right] \\ + \beta \max\{d(AM(Mz), BDz), d(AM(Mz), S(Mz)), d(BDz, S(Mz))\} \\ + \gamma \{d(BDz, Tz) + d(Tz, AM(Mz)) + d(S(Mz), BDz)\} \\ + \eta \left[ \frac{d(Tz, BDz)d(S(Mz), AM(Mz))}{d(Tz, AM(Mz)) + d(S(Mz), BDz) + d(BDz, AM(Mz))} \right]$$

$$d(Mz, z) \lesssim \alpha \left[ \frac{d(Mz, Mz) + d(z, z)d(Mz, Mz)}{1 + d(Mz, z)} \right] + \beta \max\{d(Mz, z), d(Mz, Mz), d(z, Mz)\} \\ + \gamma \{d(z, z) + d(z, Mz) + d(Mz, z)\} + \eta \left[ \frac{d(z, z)d(Mz, Mz)}{d(z, Mz) + d(Mz, z) + d(z, Mz)} \right]$$

$$d(Mz, z) \lesssim \beta d(Mz, z) + 2\gamma d(Mz, z)$$

$$d(Mz, z) \lesssim (\beta + 2\gamma)d(Mz, z) \quad \text{That is, } |d(Mz, z)| \leq (\beta + 2\gamma)|d(Mz, z)| \text{ Which is contradiction } \beta + 2\gamma < 1.$$

Therefore  $Mz = z$ . since  $AMz = z$  which implies  $Az = z$ . Now we prove that  $Dz = z$ . from (2), putting  $x = z$  and  $y = Dz$ , we get

$$d(Sz, T(Dz)) \lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BD(Dz), T(Dz))d(AMz, Sz)}{1 + d(Sz, T(Dz))} \right] \\ + \beta \max\{d(AMz, BD(Dz)), d(AMz, Sz), d(BD(Dz), Sz)\} \\ + \gamma \{d(BD(Dz), T(Dz)) + d(T(Dz), AMz) + d(Sz, BD(Dz))\} \\ + \eta \left[ \frac{d(T(Dz), BD(Dz))d(Sz, AMz)}{d(T(Dz), AMz) + d(Sz, BD(Dz)) + d(BD(Dz), AMz)} \right]$$

$$d(z, Dz) \lesssim \alpha \left[ \frac{d(z, z) + d(Dz, Dz)d(z, z)}{1 + d(z, Dz)} \right] + \beta \max\{d(z, Dz), d(z, z), d(Dz, z)\} \\ + \gamma \{d(Dz, Dz) + d(Dz, z) + d(z, Dz)\} + \eta \left[ \frac{d(Dz, Dz)d(z, z)}{d(Dz, z) + d(z, Dz) + d(Dz, z)} \right]$$

$$d(z, Dz) \lesssim \beta d(z, Dz) + 2\gamma d(z, Dz)$$

$$d(z, Dz) \lesssim (\beta + 2\gamma)d(z, Dz) \quad \text{That is, } |d(z, Dz)| \leq (\beta + 2\gamma)|d(z, Dz)| \text{ Which is contradiction } \beta + 2\gamma < 1.$$

Therefore  $Dz = z$ . since  $BDz = z$  which implies  $Bz = z$ . therefore by combining all the above result, we conclude that  $z$  is a common fixed point of  $A, B, D, M, S$  and  $T$ .

The proof is similar when  $AM$  is continuous. Similarly, the result follows when (4) holds.

**Uniqueness:** Let  $u$  be an another common fixed point of  $A, B, D, M, S$  and  $T$ . Then, we have

$$d(Sz, Tu) \lesssim \alpha \left[ \frac{d(AMz, Sz) + d(BDu, Tu)d(AMz, Sz)}{1 + d(Sz, Tu)} \right] + \beta \max\{d(AMz, BDu), d(AMz, Sz), d(BDu, Sz)\} + \\ \gamma \{d(BDu, Tu) + d(Tu, AMz) + d(Sz, BDu)\} + \eta \left[ \frac{d(Tu, BDu)d(Sz, AMz)}{d(Tu, AMz) + d(Sz, BDu) + d(BDu, AMz)} \right]$$

$$d(z, u) \lesssim \alpha \left[ \frac{d(z, z) + d(u, u)d(z, z)}{1 + d(z, u)} \right] + \beta \max\{d(z, u), d(z, z), d(u, z)\} \\ + \gamma \{d(u, u) + d(u, z) + d(z, u)\} + \eta \left[ \frac{d(u, u)d(z, z)}{d(u, z) + d(z, u) + d(u, z)} \right]$$

$$d(z, u) \lesssim \beta d(z, u) + 2\gamma d(z, u). \quad \text{Then } d(z, u) \lesssim (\beta + 2\gamma)d(z, u) \quad \text{That is, } |d(z, u)| \leq (\beta + 2\gamma)|d(z, u)|$$

Which is contradiction  $\beta + 2\gamma < 1$ . Therefore  $z = u$ . Hence  $z$  is a unique common fixed point of  $A, B, D, M, S$  and  $T$ .

**Corollary :** Let  $D, M, S$  and  $T$  be weakly reciprocally continuous self mapping of a complete complex valued metric space  $(X, d)$  satisfying conditions

1.  $S(X) \subset D(X)$  and  $T(X) \subset M(X)$
2. For each  $x, y \in X$ , where  $\alpha, \beta, \gamma$  and  $\eta$  are non negative real number with  $\alpha + \beta + 2\gamma + \eta < 1$ , such that

$$d(Sx, Ty) \lesssim \alpha \left[ \frac{d(Mx, Sx) + d(Dy, Ty)d(Mx, Sx)}{1 + d(Sx, Ty)} \right] + \beta \max\{d(Mx, Dy), d(Mx, Sx), d(Dy, Sx)\} \\ + \gamma \{d(Dy, Ty) + d(Ty, Mx) + d(Sx, Dy)\} + \eta \left[ \frac{d(Ty, Dy)d(Sx, Mx)}{d(Ty, Mx) + d(Sx, Dy) + d(Dy, Mx)} \right]$$

3. (M, S) are compatible, and M or S is continuous and (D, T) are weakly compatible.

4. (D, T) are compatible, and D or T is continuous and (M, S) are weakly compatible.

Then D, M, S and T have a unique common fixed point.

#### IV. Conclusion

In this paper we proved fixed point theorem and common fixed point theorem in complex valued metric space through concept of compatibility and continuity.

#### Acknowledgements

The authors are thankful to the referee for his/her valuable comments and suggestions.

#### References

- [1] A. Azam, B. Fisher, M. Khan, Common Fixed Point Theorems in Complex Valued Metric Spaces, *Num. Func. Anal. Opt.* 32 (2011), 243-253.
- [2] F. Rouzkard, M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Comp. Math. Appls.* 64 (2012), 1866-1874.
- [3] G. Jungck, Compatible maps and common fixed points, *Inter .J. Math. And Math. Sci.*, 9(1986), 771-779.
- [4] H. K. Nashine, M. Imdad, M. Hasan, *J. Nonlinear Sci. Appl.* 7 (2014), 42-50.
- [5] M. Imdad, J. Ali, M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, *Chaos Solitons Fractals* 42 (2009), 3121-3129.
- [6] M. Imdad, Q. H. Khan, Six mapping satisfying a rational inequality, *Radovi Math(Presently Sarajevo Journal)*, 9 (1999), 251-260.
- [7] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.* 3 (1922), 133-181.
- [8] S.M. Kang, S. Kumar, M. Kumar and P. Kumar, Common fixed point theorems for weakly compatible mappings in complex valued metric spaces. *International Journal of Pure and Applied Mathematics* 92 No. 3 2014, 403-419
- [9] V. W. Bryant, A remark on a fixed-point theorem for iterated mappings, *Amer. Math. Monthly* 75 (1968), 399-400.
- [10] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequalities Appl.* 2012 (2012), 11 pages.