



A Common Fixed Point Theorem In 2-Banach Space For Non-Expansive mapping

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Abstract: Our object in this paper to discuss about fixed point theory in 2-banach space which generalized the result of A. S. Saluja [8].

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I. Introduction

The concept of two Banach space firstly introduced by (Gahler1964) This space was subsequently been studied by mathematician (Kirk1981) and (Kirk1983) in last years. (Badshah and Gupta2005) also proved some result in 2-Banach space. (Yadav et al 2007) prove the result in 2-Banach space for non contraction mapping. (Lal and Singh1978)the analogue of Banach Contraction principle in 2-metric space for selfmap and in the present we prove a common fixed point theorem in 2-Banach Spaces by taking non-expansive mapping.

II. Preliminaries

A. Definition: [7] Let X be a real linear space and $\|\cdot, \cdot\|$ is non negative real valued function defined on X satisfying the following condition:

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent
- (ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
- (iii) $\|x, ay\| = |a|\|x, y\|$, a is real, for all $x, y \in X$.
- (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

then $\|\cdot, \cdot\|$ is called a 2-norm and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-banach space. So a 2-norm $\|x, y\|$ always satisfies $\|x, y + ax\| = \|x, y\|$ for all $x, y \in X$.

B. Definition: [7] A sequence $\{x_n\}$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is called a convergent sequence if there is a point x in X such that $\lim_{n \rightarrow \infty} \|x_n - x, a\| = 0$ for all a in X . If x_n converges to x , we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

C. Definition: [7] A sequence $\{x_n\}$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_m - x_n, a\| = 0$ for all a in X .

D. Definition: [7] A linear 2-normed space is said to be complete if every Cauchy sequence is convergent to an element on X . A complete 2-normed space X is called 2-Banach spaces.

E. Definition: [7] Let X be a 2-Banach space and T be a self mapping of X . T is said to be continuous at x if every sequence $\{x_n\}$ in X , $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$ implies $\{T(x_n)\} \rightarrow T(x)$ as $n \rightarrow \infty$.

F. Definition: [7] A function $f: R \rightarrow R$ is said to be upper semi continuous at a point $x \in R$ if given $\epsilon > 0$ there exists a neighbourhood N of x_0 in which $f(x) < f(x_0) + \epsilon$ for all $x \in N$.

G. Definition: [7] Let X be a 2-Banach space and C be a non empty bounded closed and convex subset of X . A mapping $T: C \rightarrow X$ is said to be non-expansive if $\|T(x) - T(y), a\| \leq \|x - y, a\|$ where $x, y \in C$.

III. Main Result

Theorem 3.1: Let T and G be two non expansive mapping of a 2-Banach space X into itself. T and G satisfy the following condition:

(1). $TG = GT = I$, where I is identity mapping.

$$(2). \|Tx - Gy, a\| \leq \alpha \left[\frac{\|x - Tx, a\| \|y - Gy, a\|}{\|x - y, a\|} \right] + \beta \left[\frac{\|y - Gy, a\| \|y - Tx, a\| \|x - Gy, a\| + \|x - y, a\|^3}{\|x - y, a\|^2} \right] \\ + \gamma \left[\frac{\|x - Tx, a\| + \|y - Gy, a\|}{2} \right] + \delta \left[\frac{\|x - Gy, a\| + \|y - Tx, a\|}{2} \right] + \eta \|x - y, a\|$$

Where $\alpha, \beta, \gamma, \delta, \eta \geq 0$ for all $x, y \in X$, where $4\alpha + 13\beta + 2\gamma + 2\delta + \eta \leq 2$. Then T and G have a unique common fixed point.

Proof: Suppose x is any point in 2-Banach space X .

Taking $y = \frac{1}{2} \|(T + I)x\|$, $z = G(y)$ and $u = 2y - z$, then

$$\|z - x, a\| = \|Gy - T(Gx), a\| = \|Gy - G(Tx), a\|$$

Now using (1) and (2), we have

$$\begin{aligned} \|z - x, a\| &\leq \alpha \left[\frac{\|y - Gy, a\| \|Tx - G(Tx), a\|}{\|y - Tx, a\|} \right] + \beta \left[\frac{\|Tx - G(Tx), a\| \|Tx - Gy, a\| \|y - G(Tx), a\| + \|y - Tx, a\|^3}{\|y - Tx, a\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - G(Tx), a\|}{2} \right] + \delta \left[\frac{\|y - G(Tx), a\| + \|Tx - Gy, a\|}{2} \right] + \eta \|y - Tx, a\| \\ &\leq \alpha \left[\frac{\|y - Gy, a\| \|Tx - x, a\|}{\|y - Tx, a\|} \right] + \beta \left[\frac{\|Tx - x, a\| \|Tx - Gy, a\| \|y - x, a\| + \|y - Tx, a\|^3}{\|y - Tx, a\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\|y - x, a\| + \|Tx - Gy, a\|}{2} \right] + \eta \|y - Tx, a\| \\ &\leq \alpha \left[\frac{\|y - Gy, a\| \|Tx - x, a\|}{\|y - Tx, a\|} \right] + \beta \left[\frac{\|Tx - x, a\| [\|Tx - y, a\| + \|y - Gy, a\|] \|y - x, a\| + \|y - Tx, a\|^3}{\|y - Tx, a\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\|y - x, a\| + [\|Tx - y, a\| + \|y - Gy, a\|]}{2} \right] + \eta \|y - Tx, a\| \\ &\leq \alpha \left[\frac{\|y - Gy, a\| \|Tx - x, a\|}{\left\| \frac{1}{2} (T + I)x - Tx, a \right\|} \right] \\ &\quad + \beta \left[\frac{\|Tx - x, a\| \left[\left\| Tx - \frac{1}{2} (T + I)x, a \right\| + \|y - Gy, a\| \right] \left\| \frac{1}{2} (T + I)x - x, a \right\| + \left\| \frac{1}{2} (T + I)x - Tx, a \right\|^3}{\left\| \frac{1}{2} (T + I)x - Tx, a \right\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\left\| \frac{1}{2} (T + I)x - x, a \right\| + \left\| Tx - \frac{1}{2} (T + I)x, a \right\| + \|y - Gy, a\|}{2} \right] \\ &\quad + \eta \left\| \frac{1}{2} (T + I)x - Tx, a \right\| \\ &\leq 2\alpha \|y - Gy, a\| + \beta \frac{\frac{1}{4} \|Tx - x, a\|^3 + \frac{1}{2} \|Tx - x, a\|^2 \|y - Gy, a\| + \frac{1}{8} \|Tx - x, a\|^3}{\frac{1}{4} \|Tx - x, a\|^2} \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\|Tx - x, a\| + \|y - Gy, a\|}{2} \right] + \frac{1}{2} \eta \|Tx - x, a\| \\ &\leq 2\alpha \|y - Gy, a\| + \beta \frac{\frac{3}{8} \|Tx - x, a\|^3 + \frac{1}{2} \|Tx - x, a\|^2 \|y - Gy, a\|}{\frac{1}{4} \|Tx - x, a\|^2} \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\|Tx - x, a\| + \|y - Gy, a\|}{2} \right] + \frac{1}{2} \eta \|Tx - x, a\| \\ &\leq \left(\frac{3\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx, a\| + \left(2\alpha + 2\beta \frac{\gamma}{2} + \frac{\delta}{2} \right) \|y - Gy, a\| \end{aligned} \quad (3.1)$$

Now,

$$\|u - x, a\| = \|2y - z - x, a\| = \left\| 2 \frac{1}{2} (T + I)x - Gy - x, a \right\| = \|Tx - Gy, a\| = \|Gy - Tx, a\|$$

Now using (1) and (2), we get

$$\begin{aligned} &\leq \alpha \left[\frac{\|y - Gy, a\| \|x - Tx, a\|}{\|y - x, a\|} \right] + \beta \left[\frac{\|x - Tx, a\| \|x - Gy, a\| \|y - Tx, a\| + \|y - x, a\|^3}{\|y - x, a\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|x - Tx, a\|}{2} \right] + \delta \left[\frac{\|y - Tx, a\| + \|x - Gy, a\|}{2} \right] + \eta \|y - x, a\| \\ &\leq \alpha \left[\frac{\|y - Gy, a\| \|x - Tx, a\|}{\|y - x, a\|} \right] + \\ &\quad \beta \left[\frac{\|x - Tx, a\| [\|x - y, a\| + \|y - Gy, a\|] [\|y - x, a\| + \|x - Tx, a\|] + \|y - x, a\|^3}{\|y - x, a\|^2} \right] \\ &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|x - Tx, a\|}{2} \right] + \delta \left[\frac{\|y - Tx, a\| + \|x - y, a\| + \|y - Gy, a\|}{2} \right] + \eta \|y - x, a\| \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha \left[\frac{\|y - Gy, a\| \|x - Tx, a\|}{\frac{1}{2} \|x - Tx, a\|} \right] + \beta \left[\frac{\|x - Tx, a\| \left[\frac{1}{2} \|x - Tx, a\| + \|y - Gy, a\| \right] \left[\frac{3}{2} \|x - Tx, a\| \right] + \frac{1}{8} \|x - Tx, a\|^3}{\frac{1}{4} \|Tx - x, a\|^2} \right] \\
 &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|x - Tx, a\|}{2} \right] + \delta \left[\frac{\|x - Tx, a\| + \|y - Gy, a\|}{2} \right] + \eta \frac{1}{2} \|x - Tx, a\| \\
 &\leq 2\alpha \|y - Gy, a\| + \beta \frac{\frac{3}{4} \|Tx - x, a\|^3 + \frac{3}{2} \|Tx - x, a\|^2 \|y - Gy, a\| + \frac{1}{8} \|Tx - x, a\|^3}{\frac{1}{4} \|Tx - x, a\|^2} \\
 &\quad + \gamma \left[\frac{\|y - Gy, a\| + \|Tx - x, a\|}{2} \right] + \delta \left[\frac{\|Tx - x, a\| + \|y - Gy, a\|}{2} \right] + \frac{1}{2} \eta \|Tx - x, a\| \\
 &\leq 2\alpha \|y - Gy, a\| + \beta \left[\frac{\frac{7}{8} \|x - Tx, a\|^3 + \frac{3}{2} \|x - Tx, a\|^2 \|y - Gy, a\|}{\frac{1}{4} \|Tx - x, a\|^2} \right] + \gamma \left[\frac{\|y - Gy, a\| + \|x - Tx, a\|}{2} \right] \\
 &\quad + \delta \left[\frac{\|x - Tx, a\| + \|y - Gy, a\|}{2} \right] + \eta \frac{1}{2} \|x - Tx, a\| \\
 &\leq \left(\frac{7\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx, a\| + \left(2\alpha + 6\beta + \frac{\gamma}{2} + \frac{\delta}{2} \right) \|y - Gy, a\| \quad (3.2)
 \end{aligned}$$

Now,

$$\|z - u, a\| = \|z - x, a\| + \|x - u, a\|$$

Therefore by using (3.1) and (3.2), we get

$$\|z - u, a\| \leq (5\beta + \gamma + \delta + \eta) \|x - Tx, a\| + (4\alpha + 8\beta + \gamma + \delta) \|y - Gy, a\|$$

$$\|z - u, a\| = \|Gy - 2y + z, a\| = \|Gy - 2y + Gy, a\| = 2\|y - Gy, a\|$$

$$\Rightarrow 2\|y - Gy, a\| \leq (5\beta + \gamma + \delta + \eta) \|x - Tx, a\| + (4\alpha + 8\beta + \gamma + \delta) \|y - Gy, a\|$$

$$\Rightarrow (2 - (4\alpha + 8\beta + \gamma + \delta)) \|y - Gy, a\| \leq (5\beta + \gamma + \delta + \eta) \|x - Tx, a\|$$

$$\Rightarrow \|y - Gy, a\| \leq \frac{(5\beta + \gamma + \delta + \eta)}{2 - (4\alpha + 8\beta + \gamma + \delta)} \|x - Tx, a\|$$

$$\Rightarrow \|y - Gy, a\| \leq S \|x - Tx, a\|$$

$$\text{Where, } S = \frac{(5\beta + \gamma + \delta + \eta)}{2 - (4\alpha + 8\beta + \gamma + \delta)} \leq 1, \text{ and } 4\alpha + 13\beta + 2\gamma + 2\delta + \eta \leq 2$$

Let $F = \frac{1}{2}(T + I)$, then for any $x \in X$

$$\|F^2(x) - F(x), a\| = \|F(F(x)) - F(x), a\|$$

$$\Rightarrow \|Fy - y, a\| \left\| \frac{1}{2}(T + I)y - y, a \right\| = \frac{1}{2} \|y - Ty, a\|$$

$$\Rightarrow \frac{1}{2} \|TG(y) - T(y), a\| \leq \frac{1}{2} \|Gy - y, a\|, \quad \text{because } T \text{ is nonexpansive function.}$$

$$\|F^2(x) - F(x), a\| \leq \frac{S}{2} \|x - Tx, a\|, \text{ by definition of } S.$$

We claim that $F^n(x)$ is a Cauchy sequence in X . Also by completeness $F^n(x)$ converges to $F(x)$, i.e.,

$\lim_{n \rightarrow \infty} F^n(x) = x_0 \Rightarrow T(x_0) = x_0$, therefore x_0 is fixed point of T .

$$\text{Again, } \|F^2(x) - F(x), a\| \leq \frac{S}{2} \|x - Tx, a\| = \frac{S}{2} \|TG(x) - T(x), a\| \leq \frac{S}{2} \|Gx - x, a\|$$

Because T is non expansive function. We can conclude that $G(x_0) = x_0$ that is x_0 is fixed point of G . Therefore $T(x_0) = G(x_0) = x_0$, so x_0 is a common fixed point of T and G .

Uniqueness : Let $y_0 (x_0 \neq y_0)$ be an another common fixed point of T and G . Then we have

$$\|x_0 - y_0, a\| = \|TG(x_0) - TG(y_0), a\| = \|T(Gx_0) - G(Ty_0), a\|$$

Therefore by using (2), we get

$$\begin{aligned}
 \|x_0 - y_0, a\| &\leq \alpha \left[\frac{\|Gx_0 - T(Gx_0), a\| \|Ty_0 - G(Ty_0), a\|}{\|Gx_0 - Ty_0, a\|} \right] \\
 &\quad + \beta \left[\frac{\|Ty_0 - G(Ty_0), a\| \|Ty_0 - T(Gx_0), a\| \|Gx_0 - G(Ty_0), a\| + \|Gx_0 - Ty_0, a\|^3}{\|Gx_0 - Ty_0, a\|^2} \right] + \\
 &\quad \gamma \left[\frac{\|Gx_0 - T(Gx_0), a\| + \|Ty_0 - G(Ty_0), a\|}{2} \right] + \delta \left[\frac{\|Gx_0 - G(Ty_0), a\| + \|Ty_0 - T(Gx_0), a\|}{2} \right] + \eta \|Gx_0 - Ty_0, a\| \\
 \|x_0 - y_0, a\| &\leq \alpha \left[\frac{\|x_0 - x_0, a\| \|y_0 - y_0, a\|}{\|x_0 - y_0, a\|} \right] + \beta \left[\frac{\|y_0 - y_0, a\| \|y_0 - x_0, a\| \|x_0 - y_0, a\| + \|x_0 - y_0, a\|^3}{\|x_0 - y_0, a\|^2} \right] +
 \end{aligned}$$

$$\gamma \left[\frac{\|x_0 - x_0, a\| + \|y_0 - y_0, a\|}{2} \right] + \delta \left[\frac{\|x_0 - y_0, a\| + \|y_0 - x_0, a\|}{2} \right] + \eta \|x_0 - y_0, a\|$$

$$\|x_0 - y_0, a\| \leq \beta \|x_0 - y_0, a\| + \delta \|x_0 - y_0, a\| + \eta \|x_0 - y_0, a\|$$

$$\|x_0 - y_0, a\| \leq (\beta + \delta + \eta) \|x_0 - y_0, a\|$$

Which is contradiction so $x_0 = y_0$. Hence x_0 is a unique common fixed point of T and G.

IV. Conclusion

In this paper, we have presented common fixed point theorem in 2-Banach spaces through concept of non-expansive mapping.

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Reference

- [1]. V. H. Badshah, & O.P. Gupta, Fixed point theorem in Banach and 2-Banach spaces, *Jananabha*, (2005), Vol.35,73-78.
- [2]. S. Gahlar, 2-Metreche raume and ihre topologiscche structure, *Math Nadh.*, (1964), Vol.26, 115-148.
- [3]. W.A. Kirk, Fixed point theorem for nonexpensive mappings, *Contemporary mathematics*, (1983), Vol.18,121-140.
- [4]. W.A. Kirk, Fixed point theorem for non expensive mappings, *Lecture notes in mathematics*, Springer-Verlag, Berlin and NewYork, (1981), Vol.886 ,111-120.
- [5]. S.N. Lal, & A.K. Singh, An analogous of Banach's contraction principle for 2-metric space, *Bullatin of Australian mathematical society*, (1978), Vol.18, 137-143.
- [6]. R.N. Yadva, S.S. Rajput, S. Choudhary & R.K. Bharwaj, Some Fixed point and common fixed theorem for non-contraction mapping on 2-Banach spaces, *Acta Ciencia Indica*, (2007) , Vol. 33, No.3 ,737-744.
- [7]. V. Gupta, & A.K. Tripathi, A fixed point theorems in 2-Banach space for non-expansive mapping, *Innovative System Design and Engineering*,(2011), Vol.02, No 4, 71-74.
- [8]. A.S. Saluja & A.K. Dhakde, Some fixed point and common fixed point theorems in 2-Banach space, *American Journal of Engineering Research*, (2013), Vol.02, 122-127.