



## A NOTE ON VOLUME OF PARALLELOPIPED

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**Abstract:** Determinant gives an  $n$ -dimensional volume spanned by  $n$ -independent vectors in  $\mathcal{R}^n$ . In this article we work out a method of finding the  $k$ -dimensional volume spanned by  $k$ -independent vectors in  $\mathcal{R}^n$  such that  $k \leq n$ .

**Keywords:** Determinant, Parallelopiped,  $k$ -dimensional Volume

### I. INTRODUCTION

Volume is scalar quantity associated with geometric objects. A parallelopiped is one of the simplest object in  $\mathcal{R}^n$ . More complex objects may be approximated by these parallelepiped.

**Def (Parallelepiped):** A parallelepiped  $[u_1, \dots, u_n]$  spanned by  $k$ -independent vectors  $v_1, \dots, v_k$  is the set of all points in  $\mathcal{R}^n$  satisfying following condition:

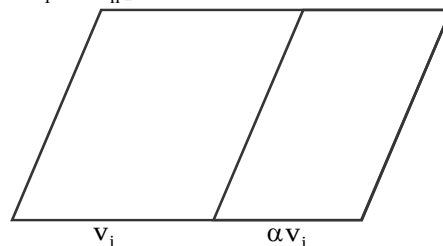
$$[v_1, \dots, v_k] := \left\{ v \in \mathcal{R}^n \mid v = \sum_{i=1}^k \alpha_i v_i \mid 0 \leq \alpha_i \leq 1 \right\}$$

Now, we can define the signed volume of parallelepiped with following properties, i.e.,

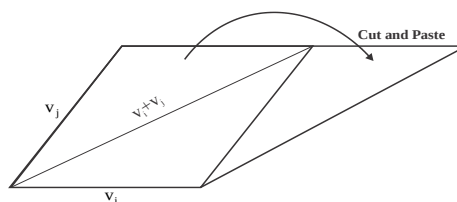
$$\text{vol}: \mathcal{R}^n \times \dots \times \mathcal{R}^n \rightarrow \mathcal{R}.$$

- I. If we multiply one side of parallelepiped with non-zero scalar  $\alpha$  then it's volume will be multiplied by same amount.

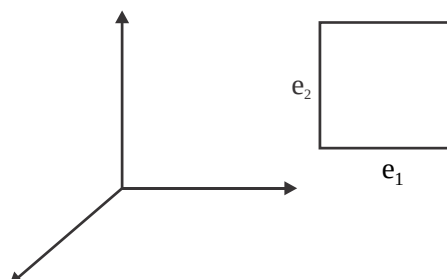
$$\text{vol}[v_1, \dots, \alpha v_i, \dots, v_n] = \alpha \text{vol}[v_1, \dots, v_i, \dots, v_n]$$



2.  $\text{vol}[v_1, \dots, v_i + v_j, \dots, v_n] = \text{vol}[v_1, \dots, v_n]$



3.  $\text{vol}[e_1, \dots, e_n] = 1$



All these properties derived from the geometric idea. To compare it with determinant we need extra property of skew-symmetry.

$$4. \quad \text{vol}[v_1, \dots, v_i, \dots, v_j, \dots, v_n] = \text{vol}[v_1, \dots, v_j, \dots, v_i, \dots, v_n]$$

Now all these property 1-4 satisfies the definition of determinant given by Curtis (1984).

$$\text{Hence, } \text{vol}([v_1, \dots, v_n]) = \det(v_1, \dots, v_n)$$

**Theorem:** Let  $W$  be some  $k$ -dimensional subspace of  $\mathfrak{R}^n$ , then there exist some orthogonal matrix such that, it transforms  $W$  to a subspace of  $\mathfrak{R}^n$  of the form  $\mathfrak{R}^k \times 0$ .

**Proof:** Choose any orthonormal basis of  $W$  such that  $b_1, \dots, b_k$ . Extend this basis to form a basis of  $\mathfrak{R}^n$ ,  $(b_1, \dots, b_n)$ . Now, Define a linear transformation  $T: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ , such that  $T(b_i) = e_i$ .

This transformation is basically change of basis. To compute the matrix associated with respect to  $T$ . Let's compute  $T^{-1}$

$$T^{-1}: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$$

$$T^{-1}(e_i) \rightarrow b_i$$

Hence,

$$B = m(T^{-1}) = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_n \\ | & | & & | \end{bmatrix}$$

$B$  is orthogonal matrix implies  $B^{-1}$  is orthogonal and the required transformation.

## II. MAIN RESULT

Volume of  $k$ -dimensional parallelepiped formed by  $k$ -independent vectors.  $x_1, \dots, x_k$  will be given by the following formula  $\text{Vol}([x_1, \dots, x_k]) = \sqrt{\det(x^T x)}$ , where  $x$  is  $n \times k$  dimensional matrix formed by columns of  $x_i$ .

**Proof:** Given vectors  $x_1, \dots, x_k, \exists A \in O(n)$ . Such that

$$Ax_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix} \text{ such that } y_i \in \mathfrak{R}^k$$

$$\text{Now, } \text{vol}([x_1, \dots, x_k]) = \det \begin{pmatrix} | & | \\ y_1 & \dots & y_k \\ | & | \end{pmatrix}_{k \times k} = \det Y$$

$$\text{Now, } (Ax)^T (Ax) = Y^T Y.$$

$$\text{Taking, } (x^T A^T A x) = (Ax)^T (Ax)$$

$$\Rightarrow x^T x = Y^T Y$$

$$\Rightarrow \det(x^T x) = \det(Y)^T \det(Y)$$

$$\Rightarrow \det(x^T x) = \det(Y) \cdot \det(Y)$$

$$= [\det(Y)]^2$$

$$\Rightarrow \det(Y) = \sqrt{\det(x^T x)}$$

$$\text{Hence, } \text{vol}[x_1, \dots, x_k] = \sqrt{\det(x^T x)}$$

## III. CONCLUSION

The volume of  $k$ -dimensional parallelepiped in terms of determinant is obtained. Determinants are relatively simple to calculate using Gaussian Elimination. Thus the volume of parallelepiped can be determined easily.

## REFERENCES

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