



A STUDY ON THE SHORTEST QUEUEING MODEL WITH JOCKEYING

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Abstract: In this paper, we consider a stochastic model of two queues (queue I and queue II) with an interconnection between them is considered. When the Principal queue length exceeds a certain number N , secondary server open his own queue. Customer arrives according to a Poisson process with rate λ . Customers arrive at a constant rate λ and join the shortest of the two queues the service time for each server is exponentially distributed but the rates may be different. Jockeying between two queues is allowed. We assume that customers that customers always join the shortest queue. Parallel queueing system is obtained.

Keywords: queueing two servers, parallel queues, retrial queues, steady-state probability

I. Introduction

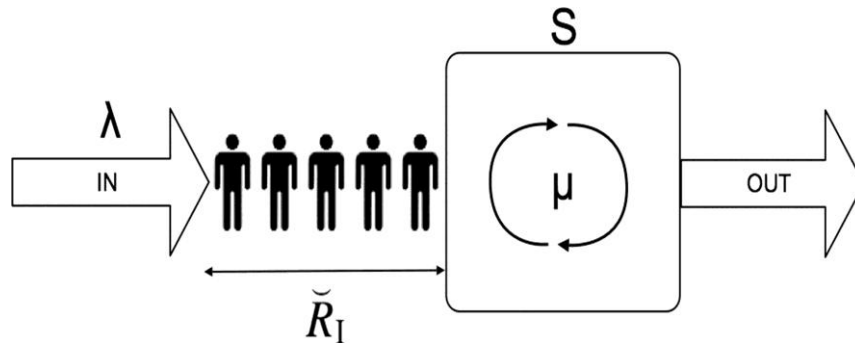
We consider many other situations; customers are often confronted with two queues, each with its own server. Customers generally enter the shortest queue available. In some situations, customers move from one queue to another. This moment is called jockeying. Such queueing systems, where customers must choose from servers, each with its own queue are called parallel queues. This concept was first studied by [5] Height (1958). His work was followed by Kingman (1961) and others. Because each server has its own queue. Several authors have studied assignment of customers to system with Parallel queues. [8] Kingman (1961) assumed that a customer would join the shortest queue available. If both queues were join the same size then a customer would join the queue. He also assumed that both servers had the same service rate. The problem to be analyzed is generally referred to as the "shortest queue problem". Zeno and Grassmann (1989) formulae based on Flatto and McKean which were superior for numerical purposes. In this paper, we consider the following modification of the shortest queue model with jockeying [9]. It is assumed that there is a Principal server who is always available when customer demand exists. When the Principal queue length exceeds a certain number N , secondary server open his own queue. Customer arrives according to a Poisson process with rate λ .

II. The Shortest Queue Model

Depending on the assumptions about the second server many models could be considered. We consider the following model upon activation of the second server; the principal queue is instantly divided between the two servers. If $N+1$ is even the two queues become equal in length, and if $N+1 = 2k+1$ is odd, we assume K customers jockey to the second server's queue. An incoming customer joins the shortest queue; or if the queues are equals in length, the customer the principal server's queue. If at the departure of a customer, one queue length exceeds the other by than one, a customer jockeys from the largest queue to the shortest line.

1. Of course several other similar models could be considered. We mention two in particulars. The second server is deactivated when the system is empty, or the secondary queue is empty and there is only one customer in the principal queue.
2. Upon activation of the second server, the lines are divided gradually customer $N+1$ joins the second server's queue. Afterwards, customers are only allowed to jockey at the completion of a service. When the two queue lengths differ by at most one, the rules for joining and jockeying are the same as for the model of this paper. We found that these cases could be readily analyzed by the methods of this paper; the quite lengthy results are not presented here.

For our model, we obtain explicit expressions for the stationary queue length distribution. We also obtain explicit expression from which the mean queue length can readily be computed. Finally, some numerical examples are presented.



III. Notation

Throughout this paper, the following notation will be used.

$\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$, $\rho = \lambda/\mu$, Where $\mu = \mu_1 + \mu_2$. In order for the stationary distribution to exist, we must assume that $\rho < 1$. We also write $\theta = \mu_2/\mu_1 = \rho_1/\rho_2$. It will be convenient to write

$$S_N = \sum_{K=0}^N \rho_1^K.$$

The S_N satisfies the simple but useful relation. $S_N = 1 + \rho S_{N-1}$, $N \geq 1$. Also, L_1 denotes the mean queue length of the principal server, L_2 denotes the mean queue length of the second server, and $L = L_1 + L_2$ is the overall mean queue length.

IV. The queue length distribution

Once the secondary service station is opened, it remains open until the system completely empty. The states for this model are as follows. The states in which there are n customers in the queue and only the principal server is active $n = 0, 1, \dots, N$. The steady-state probability of this state is denoted by q_n . The states (n, m) in which both servers are active, there are n customers in the principal queue, and there are m customers in the secondary queue,

$$|n - m| \leq 1, n + m > 0.$$

The steady-state probability for this state is denoted by P_{nm} . The difference equations for the stationary queue length distribution may be obtained by principle of stochastic balance^[11]. They are as follows.

$$\lambda q_0 = \mu_1 q_1 + \mu_1 P_{10} + \mu_2 P_{01} \quad (1)$$

$$(\lambda + \mu_1) q_n = \lambda q_{n-1} + \mu_1 q_{n+1}, 1 \leq n \leq N-1 \quad (2)$$

$$(\lambda + \mu_1) q_N = \lambda q_{N-1} \quad (3)$$

$$(\lambda + \mu_2) P_{01} = \mu_1 P_{11} \quad (4)$$

$$(\lambda + \mu_1) P_{10} = \mu_2 P_{11} \quad (5)$$

$$(\lambda + \mu) P_{nn} = \lambda P_{n-1,n} + \lambda P_{n,n-1} + \mu P_{n,n+1} + \mu P_{n+1,n}, n > 0, (n \neq K \text{ if } N = 2K-1) \quad (6)$$

$$(\lambda + \mu) P_{n,n+1} = \mu_1 P_{n+1,n+1}, n > 0 \quad (7)$$

$$(\lambda + \mu) P_{n,n-1} = \lambda P_{n-1,n-1} + \mu_2 P_{n,n}, n > 1, (n \neq K+1 \text{ if } N = 2K) \quad (8)$$

If $N = 2K-1$ then

$$(\lambda + \mu) P_{KK} = \lambda P_{K-1,K} + \lambda P_{K,K-1} + \mu P_{K+1,K} + \mu_2 P_{K,K+1} + \lambda q_N \quad (9)$$

If $N = 2K$ then

$$(\lambda + \mu) P_{K+1,K} = \lambda P_{KK} + \mu_2 P_{K+1,K+1} + \lambda q_N \quad (10)$$

From (4) and (5) we obtain

$$P_{10} = \theta \frac{\rho_1 + \theta}{\rho_1 + 1} P_{01} \quad (11)$$

We will make use of the following partial generating functions.

$$F_1(z) = \sum_{i=0}^N q_i z^i$$

$$F_2(z) = \sum_{i=0}^{\infty} P_{ii} z^i$$

$$F_3(z) = \sum_{i=0}^{\infty} P_{i,i+1} z^i$$

and

$$F_4(z) = \sum_{i=0}^{\infty} P_{i+1,i} z^i.$$

By means of the balance equations (1)-(10), we will derive the relationships between these generating functions. From (7) we obtain

$$(\lambda + \mu)F_3(z) - \mu_1 P_{01} = \frac{\mu_1}{z} F_2(z). \quad (12)$$

From (8), in the case where $N = 2K-1$, we obtain

$$(\lambda + \mu)F_4(z) - \mu_2 P_{10} = \lambda + \frac{\mu_2}{z} F_2(z). \quad (13)$$

From (8) and (10), in the case where $N = 2K$, we obtain

$$(\lambda + \mu)F_4(z) - \mu_2 P_{10} = (\lambda + \frac{\mu_2}{z})F_2(z) + \lambda q_N z^K. \quad (14)$$

From (6) and (9) we obtain

$$(\lambda + \mu)F_2(z) = (\lambda z + \mu)[F_3(z) + F_4(z)] - \mu(P_{10} + P_{01}), N = 2K. \quad (15)$$

And

$$(\lambda + \mu)F_2(z) = (\lambda z + \mu)[F_3(z) + F_4(z)] - \mu(P_{10} + P_{01}) + \lambda q_N z^K, N = 2K - 1. \quad (16)$$

We next note that since the rate of activation of the second server must equal its rate of deactivation, we have

$$\lambda q_N = \mu_1 P_{10} + \mu_2 P_{01} \quad (17)$$

Then using (1) – (3) and (17), we obtain

$$(\lambda + \mu_1 - \lambda z - \frac{\mu_1}{2})F_1(z) = [\mu_1 q_0 - \frac{\mu_1}{2} q_0 - \lambda q_N z^{N+1} + \lambda q_N]z \quad (18)$$

After some manipulation, equation (18) can be rewritten as

$$(1 - \rho_1^z)(z-1)F_1(z) = (z-1)q_0 + (P_{10} + \theta P_{01})z(1 - z^{N+1}). \quad (19)$$

From (19) we obtain

$$F_1(z) = \frac{q_0}{1 - \rho_1^z} - (P_{10} + \theta P_{01})z \frac{1 - z^{N+1}}{1 - z} \frac{1}{1 - \rho_1^z}. \quad (20)$$

From (20)

$$q_n = \rho_1^n q_0 - (P_{10} + \theta P_{01}) \sum_{k=0}^{n-1} \rho_1^k, n=1, \dots, N \quad (21)$$

Because the right hand side of (20) is a polynomial, it's also follows that

$$\rho_1^{N+1} q_0 - (P_{10} + \theta P_{01}) \sum_{k=0}^N \rho_1^k, n=0$$

And hence

$$P_{10} + \theta P_{01} = \frac{\rho_1^{N+1}}{S_N} q_0. \quad (22)$$

From (21) and (22) we obtain

$$q_n = [\rho_1^n - \rho_1^{N+1} S_{n-1}/S_N] q_0, n=1, \dots, N. \quad (23)$$

If we differentiate equation (19) and set $t = 1$, we obtain the following

$$F_1(1) = \frac{S_N - (N+1)\rho_1^{N+1}}{1 - \rho_1^{N+1}} q_0, (\rho_1 \neq 1). \quad (24)$$

By Hospital's rule, we find that

$$F_1(1) = \frac{N+2}{2} q_0, (\rho_1 = 1). \quad (25)$$

To eliminate $F_1'(1)$ we differentiate (19) twice and set $z = 1$

$$F_1'(1) = \frac{2\rho_1 S_N - 2(N+1)\rho_1^{N+2} - (N+1)(N+2)\rho_1^{N+1}(1-\rho_1)}{2(1-\rho_1)(1-\rho_1^{N+1})} q_0, (\rho_1 \neq 1). \quad (26)$$

For the case of $\rho = 1$, we must apply L'Hospital's rule to equation (26). Some tedious computations yield

$$F_1'(1) = \frac{N(N+2)}{6} q_0, (\rho_1 = 1). \quad (27)$$

From (22) and (11) we also obtain the following formulas for the two boundary probabilities P_{01} and P_{10}

$$P_{10} = \frac{(1+\rho_1)\rho_2^2\rho_1^N q_0}{(\rho_1 + \rho_2)^2 S_N} \quad (28)$$

$$P_{01} = \frac{(1+\rho_2)\rho_1^{N+2} q_0}{(\rho_1 + \rho_2)^2 S_N} \quad (29)$$

The next step is to derive a formula for the generation function $F_2(z)$. The must be done separately for the cases in which $N = 2K - 1$ is odd and in which $N = 2K$ is even. We first consider the case of $N = 2K - 1$. We solve the equations (12) and (13) for $F_3(z)$ and $F_4(z)$, respectively, and then substitute the results into equation (16). After simplification, the result is,

$$\begin{aligned} & [(\lambda + \mu)^2 z - (\lambda z + \mu)\mu_1 - (\lambda z + \mu)(\lambda z + \mu_2)] F_2(z) = \\ & (\lambda z + \mu)\mu_1 P_{01} z + (\lambda z + \mu)\mu_2 P_{10} z - (\lambda + \mu)\mu P_{01} z \\ & - (\lambda + \mu)\mu P_{10} z + (\lambda + \mu)\lambda q_N z^{K+1} \end{aligned} \quad (30)$$

After a considerable amount of computation, equation (30) can be rewritten in the more convenient form,

$$F_2(z) = \frac{(H_K(z) P_{01})}{\frac{1}{\rho^2} (1 - \frac{1}{z})(1 - \rho^2 z)} \quad (31)$$

Where $H_K(z)$ is a polynomial of the form

$$H_H(K) = Az^K + Bz - C \quad (32)$$

With

$$A = \frac{(1+\rho)(\rho_1 + \rho_2 + 2\rho_1\rho_2)}{\rho\rho_2^2(1+\rho_1)} \quad (33)$$

$$B = \frac{\rho_2^3(1+\rho_1) + \rho_1^3(1+\rho_2)}{\rho_1\rho_2(1+\rho_1)} \quad (34)$$

and

$$C = A + B = \frac{(1+\rho_2)(\rho_1 + \rho_2)}{\rho\rho_2^2} \quad (35)$$

Note that since $F_2(z)$ is analytic on a region containing the closed unit disc we must have $H_K(1)=0$. We can now use (31) to determine the P_{nn} in terms of P_{01} and hence, by virtue of (28), in terms of q_0 . A convenient expression for the p_{nn} can be obtained by using the Laurent expansion.

$$\frac{\rho^2}{(1 - \frac{1}{z})(1 - \rho^2 z)} = \sum_{n=-\infty}^{\infty} a_n z^n$$

Where

$$a_n = \rho^{2n+2} / (1 - \rho^2), n \geq 0 \text{ and } a_{-n} = \rho^2 / (1 - \rho^2), n > 0 \quad (36)$$

Then by using the form of $H_K(z)$ as given in (32) - (35) and the fact that $F_2(z)$ is analytic with $F_2(0) = 0$, we obtain the following equations,

$$P_{nn} = P_{01} [A\rho^2 + \rho^{2n} (B - C\rho^2)] / (1 - \rho^2), n=1, \dots, K, \quad (37)$$

And

$$P_{K+n, K+n} = \frac{\rho^{2K+2n+2} H_K(1/\rho^2) P_{01}}{1 - \rho^2}, n \geq 1 \quad (38)$$

From (37) we see that

$$P_{K+n, K+n} = \rho^2 P_{K+n, K+n-1} \text{ for } n \geq 2.$$

Next, from equation (7) we find that

$$P_{n, n+1} = \frac{\rho^2}{(\rho_1 + \rho_2)(1 + \rho)} P_{n+1, n+1}, n > 0 \quad (39)$$

And from (8) it follows that

$$P_{n+1, n} = \frac{\rho}{1 + \rho} \rho_{n, n} + \frac{\rho_1}{(\rho_1 + \rho_2)(1 + \rho)} P_{n+1, n+1}, n > 0 \quad (40)$$

Of course we must still determine q_0 . For this purpose we need $F_1(1)$ in terms of q_0 .

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