



A Comparative Study of Mixed Quadrature Rule with the Compound Quadrature Rules

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Abstract: A Mixed quadrature rule of higher precision for the approximate evaluation of real definite integrals has been constructed in this paper along with its analytical convergence has been studied. The relative efficiencies of the proposed mixed quadrature rule and the compound forms of Basic quadrature rules have been numerically compared by suitable test integrals and the error bound of the rule has been asymptotically estimated.

Key words: mixed quadrature rule; degree of precision; analytic function; asymptotic error estimate; error bound

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I. Introduction:

To approximate a real definite integral of the type:

$$I(f) = \int_{-1}^1 f(x)dx \quad (1.1)$$

basically there are two types of standard quadrature rules, such as:

1. Newton-Cotes type of quadrature rules and
2. Gauss- type of quadrature rules,

usually used.

It is a well-known fact that a Gaussian-type of quadrature rule numerically integrates an integral more accurately than a Newton-Cotes type of quadrature rule although both of them involve equal number of nodes. This is due to, the degree of precision of a n -point Gauss type rule (i.e. $(2n-1)$) is far more than the degree of precision of a n -point Newton-Cotes type of quadrature rule [1,6] i.e.

$$\begin{cases} n; & \text{for } n \text{ is odd} \\ (n-1); & \text{for } n \text{ is even.} \end{cases}$$

However, it is observed that when a Gauss- type of rule with certain precision (say) d is suitably mixed with Newton-cotes types of rules of same precision, a new quadrature rule of precision $(d+2)$ is produced. The resulting rule is defined as a Mixed quadrature rule [4, 5].

For example, we state here the Mixed quadrature rule:

$$R_{12}(f) = \frac{1}{5}[2R_1(f) + 3R_2(f)] \quad (1.2)$$

and

$$R_{23}(f) = \frac{1}{5}[2R_2(f) + 3R_3(f)] \quad (1.3)$$

formulated by Das and Pradhan [4] and Das and Hota [5] respectively. These rules are the weighted mean of the well-known Simpson's rules:

$$R_1(f) = \frac{1}{3}[f(-1) + 4f(0) + f(1)] \quad (1.4)$$

$$R_3(f) = \frac{1}{4}[f(-1) + 3\left\{f\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right)\right\} + f(1)] \quad (1.5)$$

of Newton-cote's type and **Gauss-Legendre 2-point rule**

$$R_2(f) = \left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \right]. \quad (1.6)$$

It is pertinent to note here that each of the quadrature rules $R_1(f)$, $R_2(f)$ and $R_3(f)$ is of precision three; whereas the mixed quadrature rules $R_{12}(f)$ and $R_{23}(f)$ are of precision five; which is two more than the degree of precision of the **Basic quadrature rules** ($R_1(f)$, $R_2(f)$ and $R_3(f)$) used to construct them.

Further it is noteworthy to mention here that the coefficients of both of the mixed quadrature rule as given in equation (1.2) and (1.3) are simple fractions: $(2/5)$ and $(3/5)$. As a result, there is no addition of errors like truncation error, round off error or machine error due to finite precision of computing machine, if the integral given in equation (1.1) is numerically integrated by this rule or by any other rules of this class of rules.

Moreover, it may not be ascertained to get the desired accuracy by applying a single quadrature rule over an integral; however the same can be achieved by employing the mixed quadrature rule up to some extent.

It is also evident that, no additional evaluation of function is required while numerically integrating the integral by a mixed quadrature rule. Again the formulation of mixed quadrature rule from the existing rules of numerical integration is quite simple but yields a rule of higher precision which produces result of greater accuracy in numerical integration.

The objective of this paper is to construct a quadrature rule of mixed type having precision nine in order to achieve higher accuracy of an integral (whose values are otherwise known) with less functional evaluations than the compound form of Basic rules used to construct the former. This fact has been numerically verified over the integrals I_1 and I_2 and the results of numerical integration are depicted in Table -4.1 to Table -4.2 in section-4 of this paper.

II. Formulation of Mixed Quadrature Rules of Precision Nine.

For the formulation of the mixed quadrature rule of precision nine, we mix the rules $R_{124}(f)$ and the rule $R_{234}(f)$ formulated by Das and Hota [5] given in equation (2.1) and (2.2) respectively. The rules as constructed by Das and Hota [5] are:

$$R_{124}(f) = \frac{1}{14} [9R_{12}(f) + 5 R_4(f)] \quad (2.1)$$

and

$$R_{234}(f) = \frac{1}{161} [81R_{23}(f) + 80 R_4(f)] \quad (2.2)$$

which are associated with the corresponding truncation errors:

$$E_{124}(f) = \frac{1}{14} [9E_{12}(f) + 5 E_4(f)] \quad (2.3)$$

and

$$E_{234}(f) = \frac{1}{161} [81E_{23}(f) + 80 E_4(f)] \quad (2.4)$$

respectively; where $R_1(f)$, $R_2(f)$ and $R_3(f)$ are given in equations (1.4) to (1.6) and the rule:

$$R_4(f) = \frac{1}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right] \quad (2.5)$$

is the Gauss-Legendre 3-point rule.

It is not to be out of place to mention here that each of the quadrature rules $R_{12}(f)$, $R_{23}(f)$ and $R_4(f)$ is of degree of precision five (i.e. same and it is the basic requirement to construct a Mixed quadrature rule) have been used to construct the degree seven mixed quadrature rules $R_{124}(f)$ and $R_{234}(f)$ by Das and Hota [5] given in equation (2.1) and (2.2) respectively. Further, by substituting formulas of the rules $R_1(f)$, $R_2(f)$ and $R_4(f)$ in the rule $R_{124}(f)$ given in equation (2.1) and then by subsequent simplification the rule $R_{124}(f)$ reduces into:

$$R_{124}(f) = \frac{3}{35} \left\{ f(-1) + \frac{208}{27} f(0) + f(1) \right\} + \frac{27}{70} \left\{ f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \right\} + \frac{25}{126} \left\{ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \right\} \quad (2.6)$$

Proceeding in the same way we obtain the rule

$$R_{234}(f) = \frac{243}{3220} \left\{ f(-1) + 3 \left\{ f\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right) \right\} + f(1) \right\} + \frac{162}{805} \left\{ f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \right\} + \frac{16}{1449} \left\{ 25 f\left(-\sqrt{\frac{3}{5}}\right) + 4 f(0) + 25 f\left(\sqrt{\frac{3}{5}}\right) \right\} \quad (2.7)$$

Now to formulate the mixed quadrature rule of precision nine we assume here that the function $f(x)$ is sufficiently differentiable in the range of integration $[-1,1]$. Then expanding $f(x)$ about $x=0$ in Taylor's series we have:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \quad (2.8)$$

where $c_n = \frac{f^{(n)}(0)}{(n)!}$; $n=1, 2, 3, \dots$ are the Taylor's coefficients.

As the series given in (2.8) is uniformly convergent in $[-1,1]$, by integrating term by term to the series (2.8), we obtain:

$$I(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \dots \quad (2.9)$$

Now by substituting $x=-1$ and $x=1$ in succession in the Taylor's expansion of $f(x)$ given in equation (2.8) we get:

$$f(-1) = c_0 - c_1 + c_2 - c_3 + \dots$$

$$f(1) = c_0 + c_1 + c_2 + c_3 + \dots$$

$\Rightarrow$

$$f(-1) + f(1) = 2c_0 + 2c_2 + 2c_4 + 2c_6 + \dots \quad (2.10)$$

Further substituting $x = \pm \frac{1}{3}, \pm \frac{1}{\sqrt{3}}$ and $x = \pm \sqrt{\frac{3}{5}}$ successively in the Taylor's expansion of $f(x)$ given in equation (2.8) we get:

$$R_{124}(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \frac{2}{7}c_6 + \frac{122}{525}c_8 + \dots \quad (2.11)$$

and

$$R_{234}(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \frac{2}{7}c_6 + \frac{8242}{36225}c_8 + \dots \quad (2.12)$$

Therefore,

$$I(f) = R_{124}(f) - \frac{16}{1575}c_8 + \dots \quad (2.13)$$

and

$$I(f) = R_{234}(f) - \frac{64}{12075}c_8 + \dots \quad (2.14)$$

Now by multiplying 12 in equation (2.13) and 23 in equation (2.14) and then subtracting the resulting series we obtain after simplification

$$I(f) = \frac{1}{11} [23 R_{234}(f) - 12 R_{124}(f)] + \frac{1}{11} [23 E_{234}(f) - 12 E_{124}(f)] \quad (2.15)$$

Here we claim that

$$I(f) \cong \frac{1}{11} [23 R_{234}(f) - 12 R_{124}(f)] \quad (2.16)$$

If we denote the right hand side of the equation (2.16) as $DR(f)$ then we obtain a new quadrature rule

$$DR(f) = \frac{1}{11} [23 R_{234}(f) - 12 R_{124}(f)] \quad (2.17)$$

associated with the truncation error

$$EDR(f) = \frac{1}{11} [23 E_{234}(f) - 12 E_{124}(f)] \quad (2.18)$$

meant for the numerical integration of the real definite integral $I(f)$ given in (1.1). We termed this rule as the **Derived Mixed Quadrature Rule**.

Degree of Precision of $DR(f)$.

Since the quadrature rules $R_{124}(f)$ and $R_{234}(f)$ are of degree of precision seven and the derived mixed quadrature rule $DR(f)$ is a fully symmetric quadrature rule; thus

$$EDR(x^i) = \frac{1}{11} [23E_{234}(x^i) - 12E_{124}(x^i)] \\ = 0;$$

for $i = 0(1)7$ and i as odd.

Further,

$$EDR(x^8) = \frac{1}{11} \left[\frac{-64}{525} + \frac{64}{525} \right] \\ = 0$$

and

$$EDR(x^{10}) = -\frac{4336}{155925} \\ \neq 0;$$

implies that the rule $DR(f)$ is of precision nine.

III. Error Analysis

In this subsection we have obtained the error bounds of the truncation errors $E_{DR}(f)$ associated with the mixed quadrature rule $DR(f)$ following the techniques due to **Lether** [8]. The error bound of the rule $DR(f)$ is given in the **Theorem-1** following to the **Lemma-1**.

Lemma: 1

If $E_{DR}(f)$ denotes the truncation error in approximation of $I(f)$ by $DR(f)$ then

$$E_{DR}(x^{2\mu}) < 0; \text{ for } \mu > 5$$

Proof:

But,

$$E_{DR}(x^{10}) = -0.02780824114 < 0.$$

Also,

$$E_{DR}(x^{12}) = -0.00838939386 < 0 \text{ and}$$

$$E_{DR}(x^{14}) = -0.00154356958 < 0$$

Further,

$$E_{DR}(x^{2\mu}) = \frac{2}{2\mu+1} - DR(x^{2\mu}) \\ = \left(\frac{2}{2\mu+1} + \frac{72}{385} \right) - \frac{243}{770} \left[1 + \frac{1}{3^{2\mu-1}} \right] - \frac{500}{693} \left(\frac{3}{5} \right)^\mu \\ < \frac{2}{2\mu+1} - \frac{9}{70} = \chi(\mu) \text{ (say)}$$

Now it is sufficient to prove that:

$$\chi(\mu) < 0; \text{ for } \mu \geq 8$$

and it is proved by method of induction.

For $\mu=8$,

$$\chi(8) = -0.01092436975 < 0$$

Let us assume that:

$$\chi(\mu) < 0, \text{ for } \mu = n.$$

Then for $\mu = n+1$,

$$\chi(n+1) = \frac{2}{2n+3} - \frac{9}{70}$$

$$= \frac{2}{2n+3} - \left[\frac{2}{2n+1} - \chi(n) \right]$$

$$= \chi(n) - \frac{4}{(2n+1)(2n+3)} < 0$$

and this completes the proof of induction. Hence, the lemma is established.

Theorem- 1

If $f(z)$ is analytic in a closed disc:

$$\Omega = \{z \in C : |z| \leq r, r > 1\}; \text{ then}$$

$$|E_{DR}(f)| \leq M(r) e_{DR}(r);$$

where

$$M(r) = \max_{|z|=r} |f(z)|$$

and

$$e_{DR}(r) = \left| r \ln\left(\frac{r+1}{r-1}\right) - \frac{1}{77} \left[\frac{99}{10} \left(\frac{r^2}{r^2-1} \right) + \frac{6561}{10} \left(\frac{r^2}{9r^2-1} \right) + \frac{2500}{9} \left(\frac{r^2}{5r^2-3} \right) + \frac{704}{45} \right] \right|$$

which $\rightarrow 0$ as $r \rightarrow \infty$.

The quantity $e_{DR}(r)$ is defined as **error constant** by Lether [8].

Proof:

It may be noted here that:

$$E_{DR}(x^{2\mu}) < 0; \text{ for } \mu \geq 6$$

which is shown in **Lemma-1**.

Let

$$f(z) = f(x); \quad \text{for } z \in [-1, 1].$$

Expanding $f(z)$ by Taylor's series expansion about $z=0$, we have

$$f(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots \quad (3.1)$$

where $b_k = \frac{f^{(k)}(0)}{(k)!}$; $k=0, 1, 2, \dots$ for $z \in [-1, 1]$ are the Taylor's coefficients.

Further, since $E_{DR}(f)$ denotes the truncation error in approximation of integral $I(f)$ by the rule $DR(f)$ i.e.

$$I(f) = DR(f) + E_{DR}(f)$$

and E_{DR} being a linear operator, we obtain from equations (3.1) the following:

$$E_{DR}(f) = \sum_{k=10}^{\infty} b_k E_{DR}(x^k) \quad (3.2)$$

where

$$E_{DR}(x^k) = \int_{-1}^1 x^k dx - DR(x^k); \text{ for } k \geq 10$$

and it is easy to show that:

$$E_{DR}(x^k) = 0, \quad k=0(1)8 \quad (3.3)$$

and for k is odd; as the rule $DR(f)$ is a fully symmetric quadrature rule.

Hence equation (3.2) further simplifies to

$$E_{DR}(f) = \sum_{\mu=5}^{\infty} b_{2\mu} E_{DR}(x^{2\mu})$$

$\Rightarrow$

$$|E_{DR}(f)| \leq \sum_{\mu=5}^{\infty} |b_{2\mu}| |E_{DR}(x^{2\mu})|$$

By **Cauchy- inequality** [3];

$$|b_{2\mu}| \leq \frac{M(r)}{r^{2\mu}}$$

Thus,

$$|E_{DR}(f)| \leq M(r) \sum_{\mu=5}^{\infty} \frac{1}{r^{2\mu}} |E_{DR}(x^{2\mu})| \quad (3.4)$$

However by **Lemma- 1**, and by following the technique due to **Lether** [8],

$$\sum_{\mu=5}^{\infty} \frac{1}{r^{2\mu}} E_{DR}(x^{2\mu}) = E_{DR} \left[\left(1 - \frac{x}{r} \right)^{-1} \right] \quad (3.5)$$

Hence from equations (3.4) and (3.5), we now have

$$|E_{DR}(f)| \leq M(r) e_{DR}(r) \quad (3.6)$$

where

$$e_{DR}(r) = E_{1234} \left[\left(1 - \frac{x}{r} \right)^{-1} \right]$$

But,

$$\left| E_{DR} \left[\left(1 - \frac{x}{r} \right)^{-1} \right] \right| = \left| r \ln \left(\frac{r+1}{r-1} \right) - \frac{1}{77} \left[\frac{99}{10} \left(\frac{r^2}{r^2-1} \right) + \frac{6561}{10} \left(\frac{r^2}{9r^2-1} \right) + \frac{2500}{9} \left(\frac{r^2}{5r^2-3} \right) + \frac{704}{45} \right] \right|$$

From the expressions of $e_{DR}(r)$ it is observed that $e_{DR}(r) \rightarrow 0$ as $r \rightarrow \infty$;

which in turn implies that, $E_{DR}(f) \rightarrow 0$ as $r \rightarrow \infty$. This completes the proof of the theorem.

Comparative Study of Error Constants:

The error constant $e_{DR}(r)$ has been evaluated for different values of $r > 1$ and the results of computation are given in **Table-3.1**. The graph of $e_{DR}(r)$ are given in **Figure-3.1** following the **Table-3.1**. It is observed from the Table-3.1 of values of error constant and the corresponding graph as well that the error constant tends to zero as r tends to infinite conforming to the fact proved analytically in the above **Theorem-1**.

Table-3.1

r	$e_{DR}(r)$
1.1	0.0686714
1.2	0.0092510
1.3	0.0021711
1.6	0.0001020
1.9	0.0000116
2.3	0.0000013
2.7	0.0000002
2.9	0.0000000

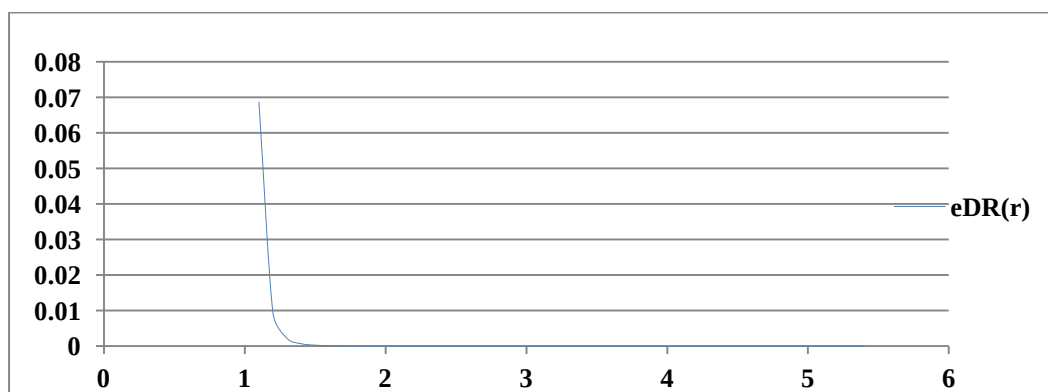


Figure-3.1

IV. Numerical Experiments

To compare the efficiency of the Mixed quadrature rule $DR(f)$ with the Compound Simpson's (1/3)rd rule, Compound Simpson's (3/8)th rule, Compound Gauss-Legendre 2-point rule and Compound Gauss-Legendre 3-point rule numerically with respect to the number of functional evaluations; we have taken the integrals:

$$I_1 = \int_{-1}^1 e^x dx \quad \text{and} \quad I_2 = \int_0^1 e^{-x^2} dx ;$$

and the result of numerical integrations is given in following Tables: Table-4.1 to Table-4.2 in order to illustrate the fact that the Mixed quadrature rule $DR(f)$ uses less number of function evaluations to reach at the desired accuracy in numerical evaluation of an integral as compared to the former.

(Notations: NF_1 -No. of functional evaluations required in Simpson's (1/3)rd rule, NF_2 -No. of functional evaluations required in Simpson's (3/8)th rule, NF_3 -No. of functional evaluations required in Gauss-Legendre 2-point rule, NF_4 -No. of functional evaluations required in Gauss-Legendre 3-point)

Table-4.1

Rules	Approx. value of I_1	Abs. Err	NF_1	NF_2	NF_3	NF_4
$R_1(f)$	2.3620538	0.01	5	-	-	-
$R_2(f)$	2.3426961	7.7×10^{-3}	-	-	2	-
$R_3(f)$	2.3556481	5.2×10^{-3}	9	4	4	-
$R_4(f)$	2.3503369	6.6×10^{-5}	13	7	8	3
$R_{23}(f)$	2.3504673	6.5×10^{-5}	17	10	12	-
$R_{12}(f)$	2.3504392	3.7×10^{-5}	21	12	12	-
$R_{124}(f)$	2.3504027	3×10^{-7}	57	33	28	6
$R_{234}(f)$	2.3504025	1×10^{-7}	69	42	44	12
$DR(f)$	2.3504024	0.0	80	72	52	18
Exact	2.3504024	**				

Table-4.2

Rules	Approx. value of I_2	Abs. Err	NF_1	NF_2	NF_3	NF_4
$R_1(f)$	0.74718043	3.6×10^{-4}	3	-	-	-
$R_2(f)$	0.74659469	2.3×10^{-4}	-	-	2	-
$R_3(f)$	0.74699232	1.7×10^{-4}	-	4	-	-
$R_4(f)$	0.74681458	9.6×10^{-6}	-	-	6	3
$R_{23}(f)$	0.74683327	9.1×10^{-6}	7	7	6	6
$R_{12}(f)$	0.74682898	4.8×10^{-6}	9	10	6	6
$R_{124}(f)$	0.74682384	2.9×10^{-7}	15	16	14	12
$R_{234}(f)$	0.74682398	1.5×10^{-7}	15	16	14	12
$DR(f)$	0.74682414	1.0×10^{-8}	35	46	44	18
Exact	*0.74682413	**				

***Correct to eight significant figures (Ref. pp.-294; Conte [2])**

V. Conclusions:

Keeping in observation to all the Tables of approximate values of the test integrals we observed that, the approximate values of the integrals I_1 and I_2 steadily increase in accuracy when any of the two integrals I_1 and I_2 are numerically integrated from rules of lowest precision to highest precision.

Thus, one can safely accept the value of an integral (whose value is not possible to get analytically) obtained corresponding to the quadrature rule of highest precision without any risk, if result accurate to seven decimal places is required in any work of science and technology.

Further, it may be noted here that a computer program for numerical evaluation of an integral can be written in such a way that the evaluation by a rule of higher precision utilizes the result of integration by two rules of lower precision involving only **three arithmetic operations: two multiplications followed by one addition**; which is a positive advantage over most popular Newton-Cote's type of rules and Gauss type of rules of very high accuracy.

From Table-4.1 and Table-4.2, it is observed from the results of numerical integrations of the integrals I_1 and I_2 that the Compound Simpson's (1/3) rd rule, Compound Simpson's (3/8)th rule, Compound Gauss-Legendre 2-point rule and Compound Gauss-Legendre 3-point rule need **89, 72, 52, 18** and **53, 46, 44, 18 number of functional evaluations** respectively to get an approximation correct up to **seven decimal figures**; whereas the same accuracy can be achieved by **Mixed quadrature rule** as formulated in this paper with **09- number of functional evaluations**. Hence Mixed quadrature rules are preferred to Compound form of **Basic rules** as former requires evaluation of function at fewer points to arrive at the result which is obtained for large values of n in case of Compound Basic rules.

References

- [1] Atkinson, Kendall. E. An introduction to numerical analysis, John Wiley and sons, 2nd edition, 1978.
- [2] Conte, S. D. & De Boor Carl. Elementary numerical analysis, Mc Graw Hill, Kogakusha, Ltd. 1972.
- [3] Conway, J. B., Functions of one complex variable, Narosa publishing house, New York. 2nd edition, 1980.
- [4] Das R. N. & Pradhan G: A mixed quadrature rule for approximate evaluation of real definite integral Int. J. Math. Educ. Sci. Technol. Vol.27, No.2, 279-283, 1996.
- [5] Das R. N. & Hota M.K, On Mixed Quadrature Rules for Numerical Integration of Real Definite Integrals, Vol. 2, No.3, pp. 109-121, 2010.
- [6] Davis, P.J & Rabinowitz, P. Methods of numerical integration, Academic press, New York, 2nd edition, 1984.
- [7] Lether, F.G., Error bound for fully symmetric cubature rules, SIAM. J. NUM. Anal.8, 36-42, 1971.