



NUMERICAL SOLUTION OF ONE - DIMENSIONAL TIME- INDEPENDENT PROBLEMS USING FEM

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Abstract: The Finite Element Method (FEM) is computational technique for numerical solution of differential equations in science and engineering. The discretization process using a finite element method is different from classical Finite Difference Method (FDM) where we replace derivatives by difference quotients. In FEM, we start from reformulation of given differential equation as an equivalent variational problem. In this paper, we focused our attention on the one - dimensional time - independent problems using FEM. The entire FEM including variational formulation, finite element formulation, choice of elements, choice of basis function, constructing stiff matrix, constructing load vector has been discussed. Numerical results for one-dimensional fourth order differential equation which arises in Euler-Bernoulli beam theory are given and discussed.

Keywords: Finite Element Method, Time Independent Problem, Cantilever Beam Problem

I. Introduction

Basic idea of any numerical method for differential equation^[1] is to discretize the given continuous problem with large degree of freedom to obtain a discrete problem or system of equation with only finite unknown that may be solved using a computer. Any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains. The Finite Element Method (FEM) is computational technique for numerical solution of differential equations in science and engineering. The technique has been used on problems^{[2]-[4]} involving *stress analysis, fluid mechanics, heat transfer, diffusion, vibrations, electrical and magnetic fields*, etc. The fundamental concept involves dividing the body under study into a finite number of subdomains called *elements*. Particular assumptions are then made on the variation of the unknown dependent variables across each element using so-called *interpolation or approximation functions*. This approximated variation is quantified in terms of solution values at special element locations called *nodes*. The discretization process using a finite element method is different from classical finite difference method, where we replace derivatives by difference quotients. Through this discretization process, the method sets up an algebraic system of equations for unknown nodal values which approximate the continuous solution^{[5]-[6]}. In FEM we start from reformulation of given differential equation as an equivalent variational problem^[7]. Because element size, shape and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading and thus this technique has become a very useful and practical tool. In Section II, the proposed algorithm is briefly summarized in six steps. In Section III, we give numerical results for *cantilever beam problem*. The FEM solution are compared with exact solution and are given in Figure1. A short summary and outlook is given in Section IV.

II. Proposed Algorithm

Here we consider the finite element formulation of one-dimensional fourth order differential equation which arises in Euler-Bernoulli beam theory. Procedure is explained in the subsequent steps using the equation which governs the transverse deflection 'w' of the beam.

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] = f(x) \quad , \quad 0 < x < L \quad (1)$$

E := modulus of elasticity

I := moment of inertia

f := transversely distributed load

Step[1]. Discretisation of Domain :

The domain of the structure (length of beam) is divided into a set of N elements (line segments) each element have

at least two nodes(end points).

Step[2]. Weak form :

The weak form of problem in solid mechanics can be developed by weighted residual method *i.e.* to multiply the equation by a test function v then integrating over the domain and finally equating to zero.

$$\int_0^L \left\{ \frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] - f \right\} v dx = 0 \quad (2)$$

$$\Rightarrow \int_0^L \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2} dx - \int_0^L v(x) f(x) dx + \left[v \frac{d}{dx} \left(EI \frac{d^2 w}{dx^2} \right) - \frac{dv}{dx} EI \frac{d^2 w}{dx^2} \right] = 0 \quad (3)$$

Variational form (3) requires that interpolation function $w(x)$ be continuous with non zero derivative up to order two. The deformation of beam must have continuous slope ($\theta = -dw/dx$), as well as continuous deflection at any two neighbouring elements. In this case, any two neighbouring beam element have common deflection and slope at shared node points. The approximation of the primary variables w, θ over finite element should be such that

$$w(x_e) = u_1^e, w(x_{e+1}) = u_3^e, \theta(x_{e+1}) = u_2^e, \theta(x_e) = u_4^e \quad (4)$$

where u_i^e is FEM solution.

Let $w(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$ then by using condition (4)

$$w^e(x) = u_1^e \phi_1^e + u_2^e \phi_2^e + u_3^e \phi_3^e + u_4^e \phi_4^e \quad (5)$$

where

$$\phi_1^e = 1 - 3 \left(\frac{x - x_e}{h_e} \right)^2 + 2 \left(\frac{x - x_e}{h_e} \right)^3, \phi_2^e = -(x - x_e) \left(1 - \frac{x - x_e}{h_e} \right)^2$$

$$\phi_3^e = 3 \left(\frac{x - x_e}{h_e} \right)^2 - 2 \left(\frac{x - x_e}{h_e} \right)^3, \phi_4^e = -(x - x_e) \left[\left(\frac{x - x_e}{h_e} \right)^2 - \left(\frac{x - x_e}{h_e} \right) \right]$$

with $x_{e+1} = x_e + h_e$, ϕ_i^e satisfies the all interpolating properties

Step[3].Finite Element Model :

The finite element model of Euler Bernoulli beam is obtained by substituting the finite element interpolation for w and ϕ_i for weight function v in weak form we get

$$\sum_{j=1}^4 \left(\int_0^L \frac{d^2 \phi_i^e}{dx^2} \frac{d^2 \phi_j^e}{dx^2} dx \right) u_j^e - \int_0^L \phi_i^e f(x) dx - Q_i^e = 0 \quad (6)$$

$$\text{where } Q_1^e = \left[\frac{d}{dx} \left(EI \frac{d^2 w}{dx^2} \right) \right]_{x_e}, Q_2^e = \left(EI \frac{d^2 w}{dx^2} \right)_{x_e}$$

$$Q_3^e = - \left[\frac{d}{dx} \left(EI \frac{d^2 w}{dx^2} \right) \right]_{x_{e+1}}, Q_4^e = - \left(EI \frac{d^2 w}{dx^2} \right)_{x_{e+1}}$$

so now we can rearrange (6) to system of equation for element 'e' as

$$[K^e][u^e] = [F^e] \quad (7)$$

$[K^e]$ is called stiffness matrix and $[F^e]$ is called load vector.

Step[4] Assembly of element matrix to global matrix :

In deriving the element equations, we isolate a typical element (eth) from the mesh and formulated the

variational problem (or weak form).

The assembly of elements is carried out by imposing two conditions:

1. Continuity of variable $u(x)$ (primary) at connecting nodes $u_i^e = u_i^{e+1}$ i.e., the last node value of element is same as the first node value of adjacent element.
2. Balance of natural variable (secondary) at connecting nodes

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied,} \\ Q_0 & \text{if an external point source } Q_0 \text{ is applied.} \end{cases}$$

Local and global nodes are connected by a matrix called connectivity matrix which comes through discretization of domain. For a mesh of E linear elements ($n = 2$) we have

$$\begin{pmatrix} K_{11}^1 & K_{11}^1 & 0 & \cdot & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^1 & K_{12}^2 & \cdot & \cdot \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & \cdot & \cdot \\ \cdot & \cdot & K_{21}^3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \cdot & K_{11}^E \end{pmatrix} \begin{pmatrix} U^1 \\ U^2 \\ \cdot \\ \cdot \\ \cdot \\ U^E \end{pmatrix} = \begin{pmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ \cdot \\ \cdot \\ \cdot \\ f_2^E \end{pmatrix} + \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ \cdot \\ \cdot \\ \cdot \\ Q_{21}^E \end{pmatrix} \quad (8)$$

This is called the assembled system.

Step[5] Imposition of boundary conditions :

One particular problem differs from other in specification of data and boundary conditions. Here we have to impose the boundary condition on the assembled set of algebraic equations.

Step[6] Post processing of solution :

If we have 2 elements then solution will be represented as

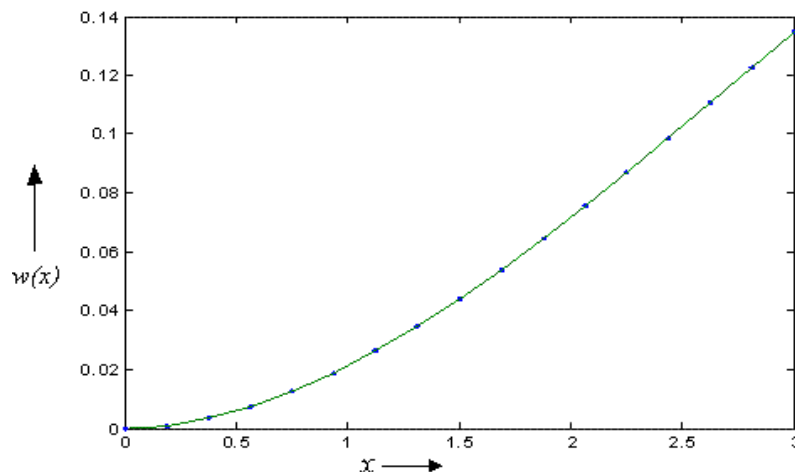
$$w^e = \begin{cases} U^3 \phi_3^1 + U^4 \phi_4^1 & 0 \leq x \leq h, \\ U^3 \phi_3^2 + U^4 \phi_4^2 + U^5 \phi_3^2 + U^6 \phi_4^2 & h \leq x \leq 2h. \end{cases} \quad (9)$$

III. Numerical Results

Consider a cantilever beam problem with 2 elements $L = 3$ m, $E.I = 5800$ kNm^2 ,

$f_0 = 24$ kNm^{-1} , $F_0 = 60$ kN, $M_0 = 0$ kNm.

To express solution w as (9), we get FEM solution that is represented by dotted curve in Figure(1) however exact solution is represented by continuous curve.



Figure(1): FEM solution (dotted curve) Vs Exact solution (continuous curve)

IV. Conclusions

The finite element methods (FEM) are having less time complexity for the complicated geometry, general boundary conditions, variable and non linear properties over finite difference method (FDM). In this case one meets unnecessary artificial complications with finite difference methodology. Further, the clear structure and versatility of finite element software makes it possible to construct general purpose software for various applications.

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