



Scientific Computations of Black-Scholes-Merton Equation for Option Pricing

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Abstract: Black-Scholes-Merton Partial differential equation represents the model for pricing an option. It is of second order parabolic type differential equation. It is a very useful application for the trading terminal. Using Black-Scholes-Merton option pricing model the trader can find the theoretical value of options (call/put). This model can also be used to price an option on a variety of assets including securities, commodities, currencies etc. It is thus important to solve Black-Scholes-Merton Partial differential equation. The solution provides fair price of an option (call/put). In the present paper several methods are discussed and we have proposed to apply Fourier transformation to solve the model with the due advantages.

Key words: Black-Scholes-Merton model, Partial differential equation, call/put options, Fourier transform method.

I. Introduction

The pricing of option is very important problem in financial market. In option pricing theory, the Black-Scholes-Merton equation is one of the most effective models for pricing options. If we consider the European call option, which gives the right to buy an asset on a specific future date, at a specific price, which depends on S-Spot price, X-Exercise Prices, t-Expiration date, r-risk free interest rate, and σ -Volatility. This model is very useful, since this requires five variables only in which four are easily available, those are S, X, t, r, and for the volatility we have to use historical data to estimate it. The formula was developed by three economists- Fisher Black, Myron Scholes and Robert Merton. They were awarded the 1997 Nobel Prize in economics for their work.

II. Theoretical Analysis

In Mathematical Finance, the Black-Scholes-Merton equation is a Partial differential equation to find the value of European Call/put option. Suppose $C(S, t)$ is the call premium then the equation,

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

is a Black-Scholes-Merton Partial differential equation.

Where S - Spot price, X - Exercise price, $t \in [0, T]$, $C(0, t) = 0$ for all t. Consider the European call option whose final payoff at the expiry time T is given by a function f of the final spot price S. $\lim_{t \rightarrow T^-} C(S, t) = f(S)$

which is continuous function need not be differentiable everywhere, as $S \rightarrow \infty$, $C(S, t) \rightarrow S$, r- risk free interest rate, σ -Volatility (both are constant).

The common assumptions found to be made in the model are:

- The options are European type
- No dividends are paid
- Movement of the market cannot be predicted
- No Commission and no transaction cost
- Interest rates are risk free and are constant
- Volatility is constant
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III. Solution of the problem

To solve the above mathematical model various approaches and methods are proposed by researcher. Some of them having limitations and some having advantages over them. In this paper different types of available

approaches are discussed and attempt is made to bring out the best suitable approach which fulfills the due advantages.

Our goal is to find the function $C(S, t) : (0, \infty) \times [0, T] \rightarrow [0, \infty)$, which satisfy the given Partial differential equation. This equation has many solutions corresponding to all the different functions that can be defined with S as the underlying variable. The particular value that is obtained when the equation is solved depends on the boundary condition. These specify the values of the call option at the boundaries of possible values of S and t . Here we will discuss several analytical methods like Laplace transform method, Separation of variable method, Fourier integral method, and Fourier transform method to solve the model.

Method (I): Use of Laplace transformation

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \quad (1)$$

With $\lim_{t \rightarrow T^-} C(S, t) = f(S)$ which is continuous function need not be differentiable everywhere.

To convert the boundary condition problem into the initial condition problem, we let

$$y = T - t$$

With this equation (1) reduce to

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC - \frac{\partial C}{\partial y} = 0 \quad (2)$$

With $\lim_{y \rightarrow 0^+} C(S, y) = f(S)$

Applying Laplace transform to equation (2) with respect to y , we get

$$\frac{1}{2} \sigma^2 S^2 \frac{d^2 \bar{C}}{dS^2} + rS \frac{d\bar{C}}{dS} - (r + q)\bar{C} = -f(S) \quad (3)$$

This is an ordinary differential equation in S .

We can find its complimentary function easily (by using solution of Cauchy-Euler equation) but it is very difficult to find particular integral, because it depends on the unknown function $f(S)$.

Method (II): Use of Variable Separable method and Fourier Integral

Second we thought to apply Variable Separable method to solve the equation. This differential equation is parabolic type differential equation so before solving this first we convert it in to one dimension Heat equation which is also of parabolic type. So in place of solving Black-Scholes-Merton Partial differential equation if we solve Heat equation then we are through. We can convert the Black-Scholes-Merton Partial differential equation to the heat equation by the following substitutions:

$$y = T - t,$$

$$x = \ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t),$$

$$D(x, y) = e^{r(T-t)} C(S, t)$$

These substitutions also convert the boundary condition

$$(\lim_{t \rightarrow T^-} C(S, t) = f(S))$$

into the initial condition $\lim_{y \rightarrow 0^+} D(x, y) = f(Xe^x)$.

Using these substitutions, we get

$$\frac{\partial D}{\partial y} = \frac{\sigma^2}{2} \frac{\partial^2 D}{\partial x^2} \quad (4)$$

With, $\lim_{y \rightarrow 0^+} D(x, y) = f(Xe^x)$

By using Variable Separable method to solve equation (4), we get following three possible solutions

$$(i) \quad D(x, y) = (C_1 e^{px} + C_2 e^{-px}) C_3 e^{\frac{\sigma^2}{2} p^2 y}$$

$$(ii) \quad D(x, y) = (C_1 \cos px + C_2 \sin px)C_3 e^{-\frac{\sigma^2}{2}p^2 y}$$

$$(iii) \quad D(x, y) = (C_1 x + C_2)C_3$$

Among the three solutions only case (ii) will give the appropriate solution which is consistent with the initial condition of the problem. Accordingly,

$$D(x, y) = (A \cos px + B \sin px) e^{-\frac{\sigma^2}{2}p^2 y} = D(x, y, p) \quad (5)$$

Now, any separable solution of the linear second order partial differential equation can be written as a linear combination of the family of solutions. As any series of function (5), found in the usual manner by taking p as multiples of a fixed number, would lead to a function that is periodic in x when $y=0$. However, since $f(Xe^x)$ is not assumed to be periodic, it is natural to use Fourier integral instead of Fourier series. Therefore, let

$$\begin{aligned} D(x, y) &= \int_0^{\infty} D(x, y, p) dp \\ &= \int_0^{\infty} (A \cos px + B \sin px) e^{-\frac{\sigma^2}{2}p^2 y} dp \end{aligned} \quad (6)$$

Now,
$$D(x, 0) = \int_0^{\infty} (A \cos px + B \sin px) dp = f(Xe^x)$$

This is Fourier integral of function f .

Where,
$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos pv \, dv \quad \text{and} \quad B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin pv \, dv$$

$$\begin{aligned} D(x, 0) &= \frac{1}{\pi} \left\{ \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(v) \cos pv \, dv \right) \cos px + \left(\int_{-\infty}^{\infty} f(v) \sin pv \, dv \right) \sin px \right] dp \right\} \\ \therefore D(x, y) &= \frac{1}{\pi} \left\{ \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(v) \cos p(x-v) e^{-\frac{\sigma^2}{2}p^2 y} \, dv \right) \right] dp \right\} \\ &= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-\frac{\sigma^2}{2}p^2 y} \cos p(x-v) \, dp \right] dv \right\} \\ &= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} f(v) \sqrt{\frac{\pi}{2s}} \frac{1}{\sigma} e^{-\frac{(x-v)^2}{2\sigma^2 y}} \, dv \right\} \end{aligned} \quad (7)$$

Taking, $Z = \frac{v-x}{\sigma\sqrt{2y}}$,
$$D(x, y) = \frac{1}{\sqrt{\pi}} \left\{ \int_{-\infty}^{\infty} f(x + \sigma Z \sqrt{2y}) e^{-Z^2} \, dZ \right\}$$

$$D(x, y) = \int_{-\infty}^{\infty} f \left(Se^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \alpha \sqrt{T-t}} \right) e^{-\frac{x^2}{2}} \, dx$$

Again taking, $x = \sqrt{2}Z$, we get
$$D(x, y) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} f(x + \sigma \sqrt{y}) e^{-\frac{x^2}{2}} \, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} f(Xe^{(x+\sigma x\sqrt{y})}) e^{-\frac{x^2}{2}} dx \right\}$$

Now substituting the values of x and y, we get

$$D(x, y) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} f(Xe^{(\ln(\frac{S}{X}) + (r - \frac{\sigma^2}{2})(T-t) + \sigma x\sqrt{T-t}))}) e^{-\frac{x^2}{2}} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} f(Se^{(r - \frac{\sigma^2}{2})(T-t) + \sigma x\sqrt{T-t}}) e^{-\frac{x^2}{2}} dx \right\}$$

This is the general solution of the model in the integral form.

Method (III): Use of Fourier transformation

Finally, we thought to use Fourier transform method to solve the model for completeness and for clarity. The Fourier transform can be viewed as an extension of the Fourier series to non-periodic functions.

Applying Fourier transform on both the sides of the equation (4), we get

$$\frac{\partial}{\partial y} F(D) + \frac{\lambda^2 \sigma^2}{2} F(D) = 0 \Rightarrow F(D) = C_1 e^{\frac{-\sigma^2 \lambda^2}{2} y}$$

Since, $D(x, 0) = f(Xe^x) \Rightarrow F(D(x, 0)) = G(\lambda)$ Where, G is the Fourier transform of f.

$$\therefore F(D) = G(\lambda) e^{\frac{-\sigma^2 \lambda^2}{2} y}$$

Taking inverse Fourier transform on both the sides we get,

$$D(x, y) = F^{-1} \left(G(\lambda) e^{\frac{-\sigma^2 \lambda^2}{2} y} \right)$$

Now, by using convolution theorem as, $F^{-1}(G(\lambda)) = f(Xe^x)$ and $F^{-1} \left(e^{\frac{-\sigma^2 \lambda^2}{2} y} \right) = \frac{1}{\sigma\sqrt{y}} e^{\frac{-x^2}{2\sigma^2 y}}$

$$D(x, y) = \int_{-\infty}^{\infty} f(v) \frac{1}{\sigma\sqrt{y}} e^{\frac{-(x-v)^2}{2\sigma^2 y}} dv$$

This is similar to what we get in equation (7). Therefore applying same steps, we get

$$C(S, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} f(Se^{(r - \frac{\sigma^2}{2})(T-t) + \sigma x\sqrt{T-t}}) e^{-\frac{x^2}{2}} dx = e^{-r(T-t)} E(f(S^*))$$

Where, $S^* = Se^{(r - \frac{\sigma^2}{2})(T-t) + \sigma x\sqrt{T-t}}$ and x is the normal random variable with mean 0 and variance 1. This is the general solution of the model in the integral form. The solution is obtained by using the condition which provides fair price of an option (call/put).

IV. Conclusion

Several methods are proposed to solve the black Scholes Merton partial differential equation. Laplace Transformation method works when we use the payoff function as $f(S) = \max\{S - X, 0\}$. Fourier analysis methods i.e., Fourier integral method and Fourier Transform method give the answer to the general payoff function $f(S)$. We are required to solve integral equation to obtain the particular solution for the model in both

the methods but general solution is obtained shortly with Fourier Transform method. Using this solution the trader can find the theoretical value of options (call/put) on a variety of assets including securities, commodities, currencies etc with different pay off functions.

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