



Almost η -Duals of Some Difference Sequence Spaces

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Abstract: Ansari and Shukla [3] have generalized the notion of α -duals and developed the concept of almost α -duals by using the concept of absolutely almost convergence. P Chandra and B.C. Tripathi [10] have introduced the concept of η -duals. Kizmaz [6] introduced the difference sequence spaces which was later on generalized by sarigöl [11] and Ahmad & Mursaleen [1] and others Ansari & Gupta [2] have introduced the concept of almost η -duals. In this paper we have determined the almost η -duals of some difference sequence spaces

Key Words: α -duals, Almost α -duals, η -duals, almost η -duals, difference sequence spaces

I. Introduction

The concept of almost convergence is introduced by Lorentz [8] and the concept of absolutely almost convergence is introduced by Das, Kuttner and Nanda [4]. Köthe - Toeplitz [7] defined α and β -duals in scalar case which was later generalized in operator version [9]. Ansari and Gupta [2] introduced the concept of almost η -duals using the concept of almost α -duals developed by Ansari and Shukla [3] and the concept of η -duals developed by P. Chandra and B.C. Tripathy [10]. The difference sequence spaces is introduced by Kizmaz [6] which was later generalized by Sarigöl [11] and Ahmad & Mursaleen [1] and others. We have determined the almost η -duals of some difference sequence space in this paper.

Some Definitions and Relations :

The sequence spaces defined by Kizmaz [6] are

$$l_{\infty}(\Delta) = \{x = (x_k) : \Delta x \in l_{\infty}\}, \quad c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$

$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\}, \text{ where } \Delta x = (\Delta x_k) = (x_k - x_{k+1})$$

Above spaces are Banach spaces with the norm $\|x\| = \|x_1\| + \|\Delta x\|_{\infty}$ where $\|\Delta x\|_{\infty} = \sup_{k \geq 2} |x_k - x_{k+1}|$. For

convenience, we denote these spaces by Δl_{∞} , Δc and Δc_0 and call these constituent sequences by Δ -bounded, Δ -convergent, Δ -null sequences, respectively.

Let E be any of the spaces l_{∞} , c and c_0 then it is easy to see that $E \subseteq \Delta E$. Further, the containment is strict. For example. Let $x_k = k$, $k = 1, 2, 3, \dots$. Then the sequence $(x_k) \notin c$ but $(x_k) \in \Delta c$. Let $p = (p_k)$ denote a sequence of strictly positive numbers (not necessarily bounded). Ahmad and Mursaleen [1] defined the sequence spaces

$$\Delta l_{\infty}(p) = \{x = (x_k) : \Delta x \in l_{\infty}(p)\}$$

$$\Delta c(p) = \{x = (x_k) : \Delta x \in c(p)\}$$

$$\Delta c_0(p) = \{x = (x_k) : \Delta x \in c_0(p)\}$$

when (p_k) is constant with all terms equal to $p > 0$, we have $\Delta l_{\infty}(p) = l_{\infty}(\Delta)$, $\Delta c(p) = c(\Delta)$ and $\Delta c_0(p) = c_0(\Delta)$.

We make use in the proof of theorem (1) and (2) of the following Lemma whose proof can be found in Kizmaz [6].

Lemma : 1. $\sup_{k \geq 1} |x_k - x_{k+1}| < \infty$ iff

$$(i) \quad \sup_{k \geq 1} k^{-1} |x_k| < \infty \quad \text{and} \quad (ii) \quad \sup_{k \geq 1} |x_k - k(k+1)^{-1} x_{k+1}| < \infty$$

Theorem : 1. The almost η -dual of $l_{\infty}(\Delta)$ is $[l_{\infty}(\Delta)]^{\hat{\eta}} = \hat{D}_r$, where

$$\hat{D}_r = \{a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right| < \infty \text{ uniformly for every } n\} \quad ..(1)$$

Proof : Let $(x_k) \in l_\infty(\Delta)$ and $a = (a_k) \in \hat{D}_r$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |\phi_{k,n}|^r &= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r = \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \frac{x_{n+i}}{(n+i)} \right|^r \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \sup_{1 \leq i \leq k} \left| \frac{x_{n+i}}{(n+i)} \right|^r \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r \\ &\leq \sup_{i \geq 1} \frac{|x_i|^r}{i^r} \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r \\ &< \infty, \text{ for every } n \text{ \{using lemma (1) and equation (1)\}} \end{aligned}$$

Thus $\sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty$, for all n . Hence, $a = (a_k) \in [l_\infty(\Delta)]^{\hat{\eta}}$. Therefore $\hat{D}_r \subset [l_\infty(\Delta)]^{\hat{\eta}}$

Conversely, let us suppose that $a = (a_k) \in [l_\infty(\Delta)]^{\hat{\eta}}$ but $a_k \notin \hat{D}_r$

$$\text{i.e. } \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r = \infty, \text{ for some } n$$

Choose $x = x_k$ with $x_k = k \operatorname{sgn} a_k$. Then $x \in l_\infty(\Delta)$

$$\begin{aligned} \text{But, } \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r &= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i a_{n+i} (n+i) (\operatorname{sgn} a_{n+i}) \right|^r \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) |a_{n+i}| \right|^r = \infty \end{aligned}$$

which is contradiction to the fact that $(a_k) \in [l_\infty(\Delta)]^{\hat{\eta}}$

Therefore $[l_\infty(\Delta)]^{\hat{\eta}} \subset \hat{D}_r$. Thus $[l_\infty(\Delta)]^{\hat{\eta}} = \hat{D}_r$. This completes the proof.

Theorem : 2. For every strictly positive sequence $p = (p_k)$, we have $[\Delta l_\infty(p)]^{\hat{\eta}} = \hat{D}_r(p)$, where

$$\hat{D}_r(p) = \bigcap_{n \geq 1} \{a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} N^{\frac{1}{p_{n+i}}} \right|^r < \infty, \text{ uniformly in } n\} \quad \dots(2)$$

Proof :

Let $a = (a_k) \in \hat{D}_r(p)$ and $x = (x_k) \in \Delta l_\infty(p)$. we choose an integer $N > \max \{1, \sup_{k \geq 1} \left| \frac{x_k}{k} \right|^{p_k}\}$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |\phi_{k,n}|^r &= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r = \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \frac{x_{n+i}}{n+i} \right|^r \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \sup_{1 \leq i \leq k} \frac{|x_{n+i}|^r}{(n+i)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r \\ &\leq \sup_{1 \leq i \leq \infty} \frac{|x_{n+i}|^r}{(n+i)^r} \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r \\ &< N^{\frac{r}{p_{n+i}}} \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \right|^r \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} \left(N^{\frac{1}{p_{n+i}}} \right) \right|^r < \infty \end{aligned}$$

Therefore $a = (a_k) \in [\Delta l_\infty(p)]^{\hat{\eta}}$ for all n (using equation (2))

Hence $\hat{D}_r(p) \subset [\Delta l_\infty(p)]^{\hat{\eta}}$.

converse, now suppose that $a = (a_k) \in [\Delta l_\infty(p)]^{\hat{\eta}}$

but $a = (a_k) \notin \hat{D}_r(p)$

$\Rightarrow$ There is an integer $N > 1$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} N^{\frac{1}{p_{n+i}}} \right|^r = \infty$$

choose $x = (x_k)$ such that $x_k = k \cdot \hat{D}_r N^{\frac{1}{p_k}} \operatorname{sgn} a_k$. Then, $x \in \Delta l_\infty(p)$ but

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right| \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} (n+i) N^{\frac{1}{p_{n+i}}} \operatorname{sgn} a_{n+i} \right| \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i) |a_{n+i}| N^{\frac{1}{p_{n+i}}} \right|^r = \infty \end{aligned}$$

which is contradiction that $a = (a_k) \in [\Delta l_\infty(p)]^{\hat{\eta}}$. Hence $[\Delta l_\infty(p)]^{\hat{\eta}} \subset \hat{D}_r(p)$

Thus, $[\Delta l_\infty(p)]^{\hat{\eta}} = \hat{D}_r(p)$ which completes the proof.

Generalized Almost η -dual of $l_\infty(\Delta^m)$: Et and colak [5] defined the sequence spaces

$$\begin{aligned} l_\infty(\Delta^m) &= \{x = (x_k) : \Delta^m x \in l_\infty\} \\ c(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c\} \\ c_0(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c_0\} \\ \text{where } m \in \mathbb{N}, \Delta^0 x &= x_k, \Delta x = (x_k - x_{k+1}) \\ \Delta^m(x) &= (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}) \end{aligned}$$

These spaces are Banach spaces with the norm

$$\|x\|_\Delta = \sum_{i=1}^m |x_i| + \|\Delta^m x\|_\infty \quad \text{where, } \|\Delta^m x\|_\infty = \sup_{k > m} |\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}|$$

In this section, we have determined the almost η -dual of $l_\infty(\Delta^m)$.

We make use in the proof of theorem [3] of the following lemmas whose proof can be found in Et and colak [5].

Lemma : 2. $\sup_{k > m} |\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}| < \infty$ iff

$$(i) \quad \sup_{k \geq 1} k^{-1} |\Delta^{m-1} x_k| < \infty \quad \text{and} \quad (ii) \quad \sup_{k \geq 1} |\Delta^{m-1} x_k - k(k+1)^{-1} \Delta^{m-1} x_{k+1}| < \infty.$$

Lemma : 3. $x \in l_\infty(\Delta^m)$ implies that $\sup_k k^{-m} |x_k| < \infty$.

Theorem : 3. The almost η -dual of $l_\infty(\Delta^m)$ is $[l_\infty(\Delta^m)]^{\hat{\eta}} = \hat{U}_r$ where

$$\hat{U}_r = \{a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r < \infty \text{ uniformly for all } n\}. \quad \dots(3)$$

Proof : Let $a = (a_k) \in \hat{U}_r$ and $x = (x_k) \in l_\infty(\Delta^m)$ Then

$$\sum_{k=1}^{\infty} |\phi_{k,n}|^r = \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r = \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \frac{x_{n+i}}{(n+i)^m} \right|^r$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \sup_{1 \leq i \leq k} \frac{|x_{n+i}|^r}{(n+i)^{rm}} \left| \sum_{i=1}^k (n+i)^m a_{n+i} \right|^r$$

$$\leq \sup_{i \geq 1} \frac{|x_i|^r}{i^{rm}} \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r < \infty \quad \{\text{By Lemma (2) and equation (3)}\}.$$

Hence $a = (a_k) \in [l_{\infty}(\Delta^m)]^{\hat{n}}$ Therefore $\hat{U}_r \subseteq [l_{\infty}(\Delta^m)]^{\hat{n}}$.

converse, Let $a = (a_k) \in [l_{\infty}(\Delta^m)]^{\hat{n}}$

Then $\sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty$ for each $x = (x_k) \in l_{\infty}(\Delta^m)$

For the sequence $x = (x_k)$ defined by

$$x_k = k^m, \quad k > m$$

$$= 0, \quad k \leq m$$

Then, $x = (x_k) \in l_{\infty}(\Delta^m)$

and hence $\sum_{k=1}^{\infty} |\phi_{k,n}|^r = \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r < \infty$.

$$\sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r$$

$$= \sum_{k=1}^m \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r + \sum_{k=m+1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r$$

$$= \sum_{k=1}^m \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r + \sum_{k=m+1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r$$

$$\leq \sum_{k=1}^m \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i)^m a_{n+i} \right|^r + \sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty$$

Hence $a = (a_k) \in \hat{U}_r$ Therefore, $[l_{\infty}(\Delta^m)]^{\hat{n}} \subset \hat{U}_r$

Thus, $[l_{\infty}(\Delta^m)]^{\hat{n}} = \hat{U}_r$

Generalized almost η -dual of $l_{\infty}(\Delta_V)$: Let $v = (v_k)$ be a sequence of non-zero complex numbers such that

$$\frac{|v_k|}{|v_{k+1}|} + 1 = 0 \quad (1/k) \text{ for each } k \quad \text{and} \quad k^{-1} |v_k| \sum_{i=1}^k |v_i^{-1}| = 0 \quad (1)$$

and $(k |v_k^{-1}|)$ is a sequence of positive numbers increasing monotonically to infinity.

Srivastava and Gnanaseelan [12] defined $F(\Delta_V) = \{x = (x_k) : (v_k(x_k - x_{k+1})) \in F\}$, where $F = l_{\infty}, c$ or c_0 obviously, these spaces are linear spaces. Further, it is easy to verify that

(i) $F(\Delta_V) \subset F(\Delta)$ if $|\Delta_k| \geq 1$ and $|v_k|$ increases to ∞ .

(ii) $F(\Delta) \subset F(\Delta_V)$ if $0 < |v_k| < 1$ and $|v_k|$ decreases to 0, These spaces are Banach spaces with the norm

$$\|x\|_{\Delta_V} = |v_1 x_1| + \sup_k |v_k(x_k - x_{k+1})|$$

In this section, we have determined the almost η -dual of $l_{\infty}(\Delta_V)$.

We make use in the proof of the theorem (4) of the following Lemma whose proof can be found in Srivastav and Gnanaseelan [4].

Lemma : 4. $\sup |v_k(x_k - x_{k+1})| < \infty$ iff

(i) $\sup_{k \geq 1} k^{-1} |v_k x_k| < \infty$ and (ii) $\sup_{k \geq 1} |v_k(x_k - k(k+1)^{-1} x_{k+1})| < \infty$.

Theorem : 4. The almost η -dual of $l_{\infty}(\Delta_V)$ is $[l_{\infty}(\Delta_V)]^{\hat{n}} = \hat{D}_V(r)$, where

$$\hat{D}_v(r) \{a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i) v_{n+1}^{-1} a_{n+i} \right|^r < \infty \text{ uniformly for every } n \text{ and } 0 < r \leq 1\} \dots (4)$$

Proof : Let $x = (x_k) \in l_{\infty}(\Delta_v)$ and $a = (a_k) \in \cdot$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |\phi_{k,n}|^r &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (v_{n+i}^{-1} a_{n+i}) \left(\frac{x_{n+i} v_{n+i}}{n+i} \right) (n+i) \right|^r \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \sup_{1 \leq i \leq k} \left| \frac{v_{n+i} x_{n+i}}{n+i} \right|^r \left| \sum_{i=1}^k i (n+i) v_{n+i}^{-1} a_{n+i} \right|^r \\ &\leq \sup_{1 \leq i \leq \infty} \left| \frac{v_{n+i} x_{n+i}}{n+i} \right|^r \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (n+i) v_{n+i}^{-1} a_{n+i} \right|^r \\ &< \sup_{i \geq 1} \left| \frac{v_i x_i}{i} \right|^r \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (n+i) v_{n+i}^{-1} a_{n+i} \right|^r \\ &< \infty \quad \{\text{using Lemma (4) and equation (4)}\} \end{aligned}$$

Thus $\sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty$ for all n . Hence $a = (a_k) \in [l_{\infty}(\Delta_v)]^{\hat{n}}$

Therefore, $D_v(r) \subset [l_{\infty}(\Delta_v)]^{\hat{n}}$.

conversely, Let us suppose that $a = (a_k) \in [l_{\infty}(\Delta_v)]^{\hat{n}}$ but $(a_k) \notin \hat{D}_v(r)$ i.e.

$$\sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i(n+i) a_{n+i} v_{n+i}^{-1} \right|^r = \infty, \text{ for some } n. \text{ choose } x = (x_k) \text{ with } x_k = k \left| v_k^{-1} \right| \text{sgn } a_k,$$

Then, $(x_k) \in l_{\infty}(\Delta_v)$. because

$$\begin{aligned} |v_k (x_k - k(k+1)^{-1} x_{k+1})| &= |v_k (k \text{sgn } a_k - k \text{sgn } a_{k+1})| \\ &= |(k \text{sgn } a_k) - (k v_{k+1}^{-1} |v_k| \text{sgn } a_{k+1})| \\ &= k \left| \text{sgn } a_k - \frac{|v_k|}{|v_{k+1}|} \text{sgn } a_{k+1} \right| \\ &= k \left| 1 + \frac{|v_k|}{|v_{k+1}|} \right| \\ &= k \cdot 0 \left(\frac{1}{k} \right) \\ &= 0(1) \end{aligned}$$

Thus $\sup_{k \geq 1} |v_k - k(k+1)^{-1} x_{k+1}| < \infty$ and

$$\text{and } \sup_{k \geq 1} k^{-1} |v_k x_k| = \sup_{k \geq 1} k^{-1} |v_k k v_k^{-1} \text{sgn } a_k| = \sup_{k \geq 1} |\text{sgn } a_k| = 1 < \infty.$$

Then by Lemma (4), we have $\sup_{k \geq 1} |v_k (x_k - x_{k+1})| < \infty$

$$\begin{aligned} \text{But } \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r \\ = \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (n+i) |v_{n+i}^{-1}| \cdot |a_{n+i}| \right|^r = \infty \end{aligned}$$

which is contradiction that $a = (a_k) \in [l_\infty(\Delta_v)]^{\hat{\eta}}$

Therefore, $[l_\infty(\Delta_v)]^{\hat{\eta}} \subset \hat{D}_v(r)$

Thus $[l_\infty(\Delta_v)]^{\hat{\eta}} = \hat{D}_v(r)$

This completes the proof.

Semi Difference sequence space and their η -dual : Let w be the space of all complex valued sequences $x = (x_k)$. Kizmaz [6] introduced the difference sequence space.

$$l_\infty(\Delta) = \{x = (x_k) \in w : (\Delta x_k)_{k=1}^\infty \in l_\infty\}$$

He further proved that a sequence $(x_k) \in l_\infty(\Delta)$

$$\text{iff} \quad (i) \quad \sup_{k \geq 1} \frac{|x_k|}{k} < \infty \quad \text{and} \quad (ii) \quad \sup_{k \geq 1} |x_k - k(k+1)^{-1} x_{k+1}| < \infty$$

By using above two conditions.

Ansari and Shukla [3] introduced a new sequence space and call it to be a semi difference sequence space. Thus

a semi difference sequence (x_k) is defined as a sequence (x_k) such that $\sup_{k \geq 1} \frac{|x_k|}{k} < \infty$ or a semi difference sequence space

$$l_\infty(S\Delta) = \{x = (x_k) \in w : \sup_{k \geq 1} \frac{|x_k|}{k} < \infty\} \quad \dots(5)$$

Ansari and Shukla [3] have proved that $l_\infty(S\Delta)$ is a Banach space with norm defined by

$$\|x\| = \sup_{k \geq 1} \frac{|x_k|}{k}$$

In this section, we have determined the almost η -dual of $l_\infty(\leq \Delta)$.

Theorem : 5. The η -dual of $l_\infty(s\Delta)$ is $l_\infty^\eta(s\Delta) = D_s(r)$ where

$$D_s(r) = \{a = (a_k) \in w : \sum_{k=1}^\infty k^r \cdot |a_k|^r < \infty \text{ and } 0 < r \leq 1\}. \quad \dots(6)$$

Proof : Let $a = (a_k) \in D_s(r)$ and $x = (x_k) \in l_\infty(s\Delta)$ Then,

$$\sum_{k=1}^\infty |a_k x_k|^r = \sum_{k=1}^\infty k^r |a_k|^r \frac{|x_k|^r}{k^r} \leq \left(\sup_{k \geq 1} \frac{|x_k|^r}{k^r} \right) \left(\sum_{k=1}^\infty k^r \cdot |a_k|^r \right) < \infty \quad \{\text{using equations (5) and (6)}\}$$

$$\text{Thus } a = (a_k) \in l_\infty^\eta(s\Delta) \quad \text{Hence } D_s(r) \subset l_\infty^\eta(s\Delta) .$$

conversely, If $a = (a_k) \in l_\infty^\eta(s\Delta)$

$$\Rightarrow \quad \sum_{k=1}^\infty |a_k x_k|^r < \infty \text{ for each } x = (x_k) \in$$

Define $x = (x_k)$, where $x_k = k$, for each k . Then $x = (x_k) \in l_\infty(s\Delta)$

$$\sum_{k=1}^\infty |a_k x_k|^r = \sum_{k=1}^\infty |a_k|^r |x_k|^r = \sum_{k=1}^\infty k^r |a_k|^r < \infty$$

Thus $a = (a_k) \in D_s(r)$ Hence $l_\infty^\eta(s\Delta) \subset D_s(r)$.

Therefore, $l_\infty^\eta(s\Delta) = D_s(r)$, which completes the proof.

Theorem : 6. The almost η -dual of $l_\infty(s\Delta)$ is $l_\infty^{\hat{\eta}}(s\Delta) = \hat{D}_s(r)$, where

$$\hat{D}_s(r) = \{a = (a_k) : \sum_{k=1}^\infty \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} \right|^r < \infty \text{ uniformly in } n, \text{ for } 0 < r \leq 1\}. \quad \dots(7)$$

Proof : Let $x = (x_k) \in l_\infty(s\Delta)$ and $a = (a_k) \in \hat{D}_s(r)$. Then,

$$\begin{aligned}
 \sum_{k=1}^{\infty} |\phi_{k,n}|^r &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r \\
 &= \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} (n+i) \frac{x_{n+i}}{n+i} \right|^r \\
 &\leq \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \sup_{1 \leq i \leq k} \left(\frac{|x_{n+i}|^r}{(n+i)^r} \right) \left| \sum_{i=1}^k i a_{n+i} (n+i) \right|^r \\
 &\leq \left(\sup_{i \geq 1} \frac{|x_i|^r}{i^r} \right) \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} (n+i) \right|^r \\
 &< \infty, \text{ for each } n, \{ \text{using equation (5) and (7)} \}
 \end{aligned}$$

Thus $\sum_{k=1}^{\infty} |\phi_{k,n}|^r < \infty$, for each n . Therefore, $\hat{D}_s(r) \subset [l_{\infty}(s\Delta)]^{\hat{n}}$

converse, Let us suppose that $a = (a_k) \in [l_{\infty}(s\Delta)]^{\hat{n}}$

But $a = (a_k) \notin \hat{D}_s(r)$ i.e. $\sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (n+i) a_{n+i} \right|^r = \infty$, for some n .

Choose $x = (x_k)$ with $x_k = k$, for each k . Then $x = (x_k) \in l_{\infty}(s\Delta)$

$$\begin{aligned}
 \text{but } \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i a_{n+i} x_{n+i} \right|^r \\
 = \sum_{k=1}^{\infty} \frac{1}{k^r (k+1)^r} \left| \sum_{i=1}^k i (n+i) a_{n+i} \right|^r = \infty
 \end{aligned}$$

which is contradiction to the fact that $a = (a_k) \in [l_{\infty}(s\Delta)]^{\hat{n}}$

Hence, $[l_{\infty}(s\Delta)]^{\hat{n}} \subset \hat{D}_s(r)$. Thus $[l_{\infty}(s\Delta)]^{\hat{n}} = \hat{D}_s(r)$.

which completes the proof.

References

- [1] T. Balasubramanim and A. Pandiarari ,2009: Duals of vector sequence spaces; Int. journal of Math. Analysis, Vol. 3, no. 16, 779-785.
- [2] Köthe G. and Toeplitz O., 1934 : Lineare Raime mit unendlichvielen koordinaten and Ringe unendlicher materizen J. F. reine u. angew Math., 174 193-226.
- [3] R. G. Cooke 1949: Infinite matrices and sequence spaces Macmillan and Co. London.
- [4] G. H. Hardy 1949: Divergent series, Oxford University Press,
- [5] J. Maddox , 1970: Element of functional Analysis. Combridge University Press.
- [6] G. Köthe , 1969: Topological vector space I (English); springer-verlag.
- [7] A. R. Robinson 1995: On functional transformations and summability, Poc. London Math. Soc. 52 ,132-160
- [8] J. Maddox 1986: Infinite matrices of operators, Lecture notes in mathematics, Springer-verlag, Barlin