



Positivity Preserving Monotonic Quadratic Trigonometric Beta-Spline

Mridula Dube^a and Meenal Priya Singh^b

^a Department of Mathematics, R.D. University, Jabalpur, India

^b Christ Church School for boys and girls, ISC, Jabalpur, India

Abstract: The aim of this paper is to construct a quadratic trigonometric beta-spline to preserve the positivity and monotonicity of interpolating curve. GC^2 continuity and approximation property is discussed.

Keywords: trigonometric spline; shape parameter; interpolation; monotonic; positivity; approximation.

I. Introduction

Shape designing is one of the main area of research work. Recently research trend is concentrated on trigonometric spline, B-spline, beta spline, rational spline [1,5,8,10]. In computer graphics and engineering design shape parameters are playing a very important role [6,7,8]. The trigonometric splines were introduced by Schoenberg (1964). Now it is the spotlight for surfaces and shapes. Beta spline is a class of piecewise polynomial splines with applications in the construction of curves and surfaces for computer graphics and computer graphical geometrical design. In curve designing trigonometric polynomial is playing an important role. Its positivity and monotonicity [1,2,5,9] is important in Medical Science, Physics and Engineering designing. Barsky worked on it and a numerous articles on Beta splines and geometrical continuity have appeared in the literature. This paper is based on the construction of a quadratic spline interpolation. We have discussed the positive and monotonic condition and GC^2 continuity of a quadratic trigonometric spline. The paper is arranged as follows. In Section 2, trigonometric spline is constructed. GC^2 continuity of the rational cubic spline is discussed in section 3. In section 4, positivity is discussed. In section 5, monotonic condition is derived. Section 6 is about the approximation properties of such trigonometric spline. In section 7, numerical example and conclusion of the paper is discussed in section 8.

II. Trigonometric Spline Interpolation

In this section, the quadratic trigonometric spline is defined as, let $a = t_0 < t_1 < \dots < t_n = b$ are the knots f_i and d_i for $i = 0, 1, 2, 3, \dots, n$ are the function values and derivatives by the Arithmetic Mean method.

Quadratic trigonometric spline defined over the subinterval $[t_i, t_{i+1}]$, let $\theta_i = \frac{(t-t_i)}{h_i}$ where

$$h_i = t_{i+1} - t_i$$

$$\begin{aligned} \text{Now consider, } P_i(t) = & \left(1 - \sin \frac{\pi \theta}{2}\right)^2 A_i + 2 \sin \frac{\pi \theta}{2} \left(1 - \sin \frac{\pi \theta}{2}\right) B_i \\ & + 2 \cos \frac{\pi \theta}{2} \left(1 - \cos \frac{\pi \theta}{2}\right) C_i + \left(1 - \cos \frac{\pi \theta}{2}\right)^2 D_i \end{aligned}$$

Which satisfying the following interpolating conditions

$$p(t_i) = f_i, p(t_{i+1}) = f_{i+1}$$

and

$$p'(t_i^-) = \alpha_i d_i, p'(t_{i+1}^-) = d_{i+1}$$

After some calculation we have

$$\begin{aligned} A_i &= f_i \text{ and } D_i = f_{i+1} \\ B_i &= f_i + \frac{d_i h_i \alpha_i}{\pi} \\ \text{and} \\ C_i &= f_{i+1} - \frac{d_{i+1} h_i}{\pi} \end{aligned}$$

which gives,

$$P_i(t) = \left(1 - \sin \frac{\pi \theta}{2}\right)^2 f_i + 2 \sin \frac{\pi \theta}{2} \left(1 - \sin \frac{\pi \theta}{2}\right) \left(f_i + \frac{d_i h_i \alpha_i}{\pi}\right)$$

$$+2 \cos \frac{\pi^{\theta}}{2} \left(1 - \cos \frac{\pi^{\theta}}{2}\right) \left(f_{i+1} - \frac{d_{i+1}h_i}{\pi}\right) + \left(1 - \cos \frac{\pi^{\theta}}{2}\right)^2 f_{i+1} \quad (1)$$

Arithmetic Mean Method

This method is used to calculate the value of d_i . It is borrowed from Hussain and Sarfaraz (10). This method is the three point difference approximation with where,

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} \text{ or } \Delta_i = 0 \\ d_i^* & \text{otherwise, } i = 2, 3, \dots, n-1 \end{cases}$$

And the end conditions are given as,

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } \operatorname{sgn}(d_1^*) \neq \operatorname{sgn}(\Delta_1) \\ d_1^* & \text{otherwise} \end{cases}$$

$$d_n = \begin{cases} 0 & \text{if } \Delta_{n-1} = 0 \text{ or } \operatorname{sgn}(d_n^*) \neq \operatorname{sgn}(\Delta_{n-1}) \\ d_n^* & \text{otherwise} \end{cases}$$

Where

$$d_i^* = \left(\frac{h_i \Delta_{i-1} + h_{i-1} \Delta_i}{h_i + h_{i-1}} \right)$$

$$d_1^* = \Delta_1 + \left(\frac{h_i (\Delta_1 - \Delta_2)}{h_i + h_{i-1}} \right)$$

$$d_n^* = \Delta_{n-1} + \left(\frac{h_{n-1} (\Delta_{n-1} - \Delta_{n-2})}{h_{n-1} + h_{n-2}} \right)$$

III. GC^2 trigonometric spline

Furthermore this quadratic trigonometric spline could even be GC^2 in the interpolating interval $\{t_0, t_n\}$. We apply the following GC^2 spline interpolation on $P(t)$ define in (1) for data set $\{t_i, f_i\}$

$$P^n(t_i+) = \alpha_i^2 P^n(t_i-), i = 1, 2, \dots, n-1,$$

Thus we have system of linear equations:

$$\begin{aligned} & h_{i-1}^2 \left[f_{i+1} - f_i - \frac{2\alpha_i d_i h_i}{\pi} - \frac{d_{i+1} h_i}{\pi} \right] \\ &= h_i^2 \alpha_i^2 \left[f_{i-1} - f_i - \frac{d_{i-1} h_{i-1}}{\pi} - \frac{2d_i h_{i-1}}{\pi} \right] \\ \frac{1}{h_i \pi} d_{i+1} + \left[\frac{2\alpha_i^2}{h_{i-1} \pi} + \frac{2\alpha_i}{h_i \pi} \right] d_i + \frac{\alpha_{i-1} \alpha_i^2}{h_{i-1} \pi} d_{i-1} &= \frac{f_{i+1}}{h_i^2} + \left[\frac{\alpha_i^2}{h_{i-1}^2} - \frac{1}{h_i^2} \right] f_i - \frac{\alpha_i^2}{h_{i-1}^2} f_{i-1} \end{aligned} \quad (2)$$

$$i = 1, 2, \dots, n-1$$

We observe that the coefficient of d_{i-1} , and d_{i+1} are positive since $h_i, \alpha_i > 0$ and the coefficients of matrix is diagonal dominant, which gives

$$\left(\frac{2\alpha_i^2}{\pi h_i} + \frac{2\alpha_i}{\pi h_i} \right) - \left(\frac{1}{h_i \pi} + \frac{\alpha_{i-1} \alpha_i^2}{h_i \pi} \right) \geq 0 \quad (3)$$

after some calculation

$$\frac{1}{2} \leq \alpha_i \leq 2$$

IV. Positive spline interpolation

For positive spline interpolation the data set is (t_i, f_i) and $P(t)$ is define in (1). Data set is define as

$$t_1 < t_2 < t_3 < \dots < t_n$$

and

$$f_i > 0, i = 1, 2, 3, \dots, n$$

interpolation define in (1) is positive in other word $P(t) > 0$

if

$$\alpha_i > \frac{-f_i \pi}{d_i h_i}$$

and

$$f_{i+1} - \frac{d_{i+1} h_i}{\pi} > 0$$

Theorem 1. The quadratic trigonometric polynomial $P(t)$ define over the interval $[a, b]$ is positive if in each subinterval $[t_i, t_{i+1}]$ for positive data set (t_i, f_i) and $\alpha_i \in \left[\frac{1}{2}, 2\right]$ satisfies following conditions

$$\alpha_i > \frac{-f_i \pi}{d_i h_i}$$

and

$$f_{i+1} - \frac{d_{i+1}h_i}{\pi} > 0$$

V. Monotone spline interpolation

To get the condition for the interpolation to keep monotone in the interpolating interval we shall consider f_i is monotonic which gives,

$$f_1 \leq f_2 \leq \dots \leq f_n$$

in other words $\Delta_i \geq 0$, for $i = 1, 2, \dots, n-1$. which gives $d_i \geq 0$

In a similar fashion one can deal with a monotonic decreasing data. For $i = 1, 2, \dots, n$.

Now $P(t)$ define in (1) is monotonic increasing if its derivative is positive i.e., $P'(t) \geq 0$

$$P'_i(t) = \frac{\pi}{2} \left[2 \frac{d_i h_i \alpha_i}{\pi} \cos \frac{\pi \theta}{2} + 2 \frac{d_{i+1} h_i}{\pi} \sin \frac{\pi \theta}{2} + 2 \left(\Delta_i h_i - 2 \frac{h_i}{\pi} (d_i \alpha_i + d_{i+1}) \right) \sin \frac{\pi \theta}{2} \cos \frac{\pi \theta}{2} \right] \geq 0 \quad (4)$$

it shows the sufficient condition

$$\frac{\pi}{2d_i} \Delta_i - \frac{d_{i+1}}{d_i} \geq \alpha_i$$

Theorem 2 $P(t)$ which is define in (1) for monotonic data set (t_i, f_i) is monotonic in interval $[t_i, t_{i+1}]$ for $i = 1, 2, 3, \dots, n$ if it satisfies following condition

$$\frac{\pi}{2d_i} \Delta_i - \frac{d_{i+1}}{d_i} \geq \alpha_i$$

where $\alpha_i \in \left[\frac{1}{2}, 2 \right]$

VI. Numerical example

In this section given example shows that interpolation define in (1) is positive and monotonic under the conditions define in Theorem (1) and (2). Fig 1 is the curve generated by data given in following table. It shows the positive but not the monotonicity.

t_i	$f(t_i)$	α_i
1	3	0.5
2	6	0.9
3	5	0.99
4	8	1.0
5	1	0.5

Table 1. data for Positive graph of $P(t)$

Figure 2 is the improved from, curve is positive as well as monotonic also under the conditions discussed above.

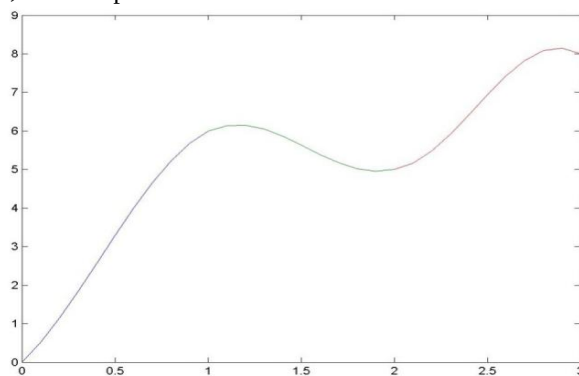


Figure 1: Positive graph of $P(t)$

t_i	$f(t_i)$	α_i
1	0	0.5
2	6	0.9
3	10	0.99
4	15	1.0
5	21	0.5

Table 2. data for Monotonic and Positive graph of $P(t)$

VII. Approximation properties of the interpolation

To Find the error estimation we consider that the given function $f(t) \in GC^2$ and $P(t)$ is the interpolating function of $f(t)$ in $[t_i, t_{i+1}]$ for $i = 0, 1, \dots, n$, consider the case that the knots are equally spaced, namely, $h_i = h = \frac{t_n - t_0}{n}$ for all $i = 1, 2, \dots, n$, using the Peano- Kernel Theorem [12] gives the following

$R[F] = f(t) - p(t) = \int f''(\tau) R_t[(t - \tau)_+] d\tau, t \in [t_i, t_{i+1}]$,
where,

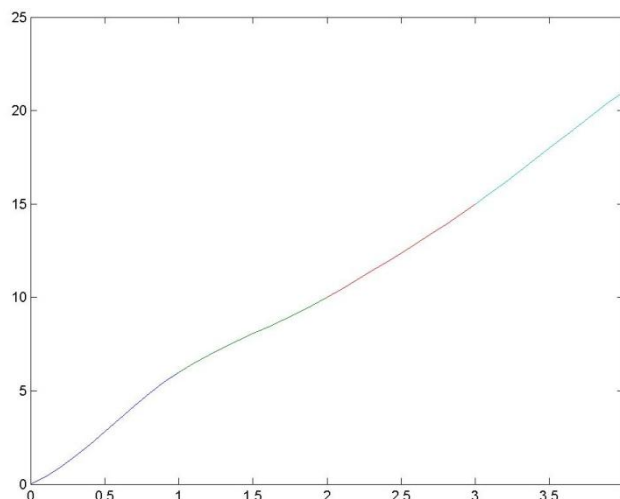


Figure 2: Monotonic and Positive graph

$$R_t[(t - \tau)_+] = \begin{cases} (t - \tau) - N(t_{i+1} - \tau) + \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) h_i, & t_i < \tau < t \\ -N(t_{i+1} - \tau) + \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) h_i, & t < \tau < t_{i+1} \\ \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) (t_{i+2} - \tau), & t_{i+1} < \tau < t_{i+2} \end{cases}$$

$$= \begin{cases} p(\tau), & t_i < \tau < t \\ q(\tau), & t < \tau < t_{i+1} \\ r(\tau), & t_{i+1} < \tau < t_{i+2} \end{cases}$$

$$N = \frac{\alpha_i}{\pi} \sin \frac{\pi\theta}{2} \left(1 - \sin \frac{\pi\theta}{2}\right) + \sin^2 \frac{\pi\theta}{2}$$

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^{(2)}(t)\| \left[\int_{t_i}^t |p(\tau)| d\tau + \int_t^{t_{i+1}} |q(\tau)| d\tau + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)| d\tau \right] \quad (5)$$

Next, we calculate the following terms in (5)

$$\int_{t_{i+1}}^{t_{i+2}} |r(\tau)| d\tau = \frac{1}{2\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) h_i^2 = h_i^2 M_1$$

We observe that for

$$q(t) = -\frac{h_i}{\pi} \left[\pi N(1 - \theta) - \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) \right] \leq 0$$

and at $\tau = t_{i+1}$

$$q(t_{i+1}) = \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) h_i \geq 0$$

It is easy to see that the root of $q(\tau)$ is

$$\tau_1^* = t_{i+1} - \frac{\cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) h_i}{N\pi}$$

Thus,

$$\int_t^{t_{i+1}} |q(\tau)| d\tau = \int_t^{\tau_1^*} -q(\tau) d\tau + \int_{\tau_1^*}^{t_{i+1}} q(\tau) d\tau$$

$$\int_t^{t_{i+1}} q(\tau) d\tau = N \left[\frac{(1 - \theta)^2 h_i^2}{2} - (K h_i)^2 \right] + \frac{1}{\pi} \cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right) [2K - (1 - \theta)] h_i^2 = h_i^2 M_2$$

where,

$$K = \frac{\cos \frac{\pi\theta}{2} \left(1 - \cos \frac{\pi\theta}{2}\right)}{N\pi}$$

Similarly, since

$$p(\tau) = (t - \tau) + q(\tau)$$

$$p(t) = q(t) \leq 0, p(t_i) = (t - t_i) + q(t_i) \geq 0$$

So there must be root of p τ called τ_1^* as given below

$$\tau_2^* = t_{i+1} + h_i \frac{(1 - \theta) - \frac{1}{\Pi} \cos \frac{\Pi\theta}{2} \left(1 - \cos \frac{\Pi\theta}{2}\right)}{N - 1}$$

we can write it

$$\tau_2^* = t_{i+1} - h_i Q$$

where

$$Q = \frac{(1 - \theta) - \frac{1}{\Pi} \cos \frac{\Pi\theta}{2} \left(1 - \cos \frac{\Pi\theta}{2}\right)}{N - 1}$$

$$\int_{t_i}^t |p(\tau)| d\tau = \int_t^{\tau_2^*} -p(\tau) d\tau + \int_{\tau_2^*}^t p(\tau) d\tau$$

$$\begin{aligned} \int_t^{t_{i+1}} |p(\tau)| d\tau &= [(1 - \theta)h_i + Q]^2 - \frac{(\theta h_i)^2}{2} + \frac{N h_i^2}{2} [(1 - \theta)^2 + \theta^2 - 2Q^2] \\ &\quad - \frac{1}{\Pi} \cos \frac{\Pi\theta}{2} \left(1 - \cos \frac{\Pi\theta}{2}\right) h_i^2 [1 - 2Q] = h_i^2 M_3 \end{aligned}$$

$$M = M_1 + M_2 + M_3$$

from the above calculations it can be shown that

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^{(2)}(t)\| h^2 M \quad (6)$$

where M is depending upon θ .

VIII. Conclusion

In this paper, a quadratic trigonometric Beta spline was constructed. It was C^2 continuous in interpolating interval. Condition for Positivity and Monotonicity derived which is useful in the field of Engineering and Medical Science.

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Acknowledgments

The authors are thankful to the referees for their helpful and valuable suggestions.