



On Applied Mathematical Programming Techniques for Estimating Frontier Production Functions

T. Suneetha, J. Prabhakara Naik, R. Md. Mastan Shareef* and R. Abbaiah

Department of Statistics, S. V. University, Tirupati-517 502, INDIA

*Associate Professor, SANA Engineering College, Kodada (A.P), INDIA

Abstract: Several authors worked on the estimation of parametric frontier production functions, as characterized by the work of Aigner and Chu(1968). Afriat (1972) and Richmond (1974), begins by assuming a function giving maximum possible output as a function of certain inputs.

Aigner and Chu (1968) suggest the estimation of the considered model by mathematical programming methods. One problem with these approaches is extreme sensitivity to outliers. . This has led to the development of so-called "probabilistic" frontiers (Timmer (1971), Dugger (1974)), which are estimated by the same types of mathematical programming techniques discussed above, except that some specified proportion of the observations is allowed to lie above the frontier. Another problem with the mathematical programming techniques is that they do not lead to estimates with known statistical properties. Indeed, these are not really statistical techniques. In an attempt to give them a statistical basis, Schmidt (1976) has added a one-sided disturbance to the considered model.

In particular, the assumption that disturbance has an exponential distribution leads to the linear programming techniques, while the assumption that disturbance has a half-normal distribution leads to the quadratic programming techniques. Therefore, Aigner and chu's estimates can be viewed as maximum likelihood estimates under particular error specifications.

In another paper, Aigner Amemiya, and Poirier (1976) construct a more reasonable error structure than a purely one-sided one.

Their justification for this error specification is that firms are presumed to differ in their "production" of y for a given set of values for the "input" according to random variation in the model their ability to utilize "best practices" technology a source of error that is one-sided.

A primary contribution of this error structure to the literature is that it allows the placement of the fitted function to be estimated along with the usual parameters of interest through the parameter θ . Thus, the criticism levied at the average function by proponents of the frontier [e.g., Aigner and chu (1968)] and criticisms that accompany strict use of the frontier or envelope function as the "appropriate" industry production function (Timmer (1971)) are ameliorated by this more accommodating specification.

Key Words:Mathematical Programming, Frontier Production Function, timation of Production Function, Stochastic Frontier Model, Production Function, Cobb-Douglas Function.

I. MATHEMATICAL PROGRAMMING FOR ESTIMATION OF PRODUCTION FUNCTION

Previous work on the estimation of parametric frontier production functions, as characterized by the work of Aigner and Cher (1968). Afriat (1972) and Richmond (1974), begins by assuming a function giving maximum possible output as a function of certain inputs. In deterministic terms, we write

$$y_i = f(\underline{x}_i; \underline{\beta}), \quad i=1,2,-----,n. \quad (1.1)$$

here y_i is output, $\underline{x}_i$ is a vector of inputs, and $\underline{\beta}$ is a parameter vector to be estimated. In the present context the N observations typically represent a cross-section of firms with in a given industry [cf., Aigner and Chu (1968)].

Aigner and Chu (1968) suggest the estimation of (1.1) by mathematical programming methods.

Specifically, they suggest minimization of

$$\sum_i |y_i - f(\underline{x}_i; \underline{\beta})|$$

Subject to $y_i \leq f(\underline{x}_i; \underline{\beta})$, which is a linear programming problem if the production function is linear. Alternatively, they suggest minimization of

$$\sum_i [y_i - f(\underline{X}_i; \underline{\beta})]^2$$

Subject to the same constraint, which is a quadratic programming problem if the production function is linear.

One problem with these approaches is extreme sensitivity to outliers. This has led to the development of so-called “probabilistic” frontiers (Timmer (1971), Dugger (1974)), which are estimated by the same types of mathematical programming techniques discussed above, except that some specified proportion of the observations is allowed to lie above the frontier. The selection of this proportion is essentially arbitrary, lacking explicit economic or statistical justification. Another problem involves reconciling the observations above the frontier with the concept of the frontier as maximum possible output. Typically this is accomplished by appealing to measurement error in the extreme observations, however, it seems preferable to incorporate the possibility of measurement error, and of other unobservable shocks, in a less arbitrary fashion.

Another problem with the mathematical programming techniques is that they do not lead to estimates with known statistical properties. Indeed, these are not really statistical techniques. In an attempt to give them a statistical basis, Schmidt (1976) has added a one-sided disturbance to (1.1) which yields the model

$$y_i = f(\underline{x}_i; \underline{\beta}) + \epsilon_i \quad (1.2)$$

Where $\epsilon_i \leq 0$. Given a distribution assumption for the disturbance term, the model can then be estimated by maximum likelihood techniques. In particular, the assumption that $-\epsilon_i$ has an exponential distribution leads to the linear programming techniques, while the assumption that $-\epsilon_i$ has a half-normal distribution leads to the quadratic programming techniques. Therefore, Aigner and chu’s estimates can be viewed as maximum likelihood estimates under particular error specifications.

Unfortunately, the observation that the model can be estimated by maximum likelihood techniques, and that under appropriate assumptions linear and quadratic programming are maximum likelihood techniques, is of little practical value.

This is so because the usual “regularity conditions” for the application of maximum likelihood are violated. In particular, since $y_i \leq f(\underline{x}_i; \underline{\beta})$, the range of the random variable y depends on the parameters to be estimated.

Therefore, the usual theorems cannot be invoked to determine the asymptotic distributions of parameter estimates. Under these circumstances it is not clear just how much we know about the frontier after having estimated it.

In another recent paper, Aigner Amemiya, and Poirier (1976) construct a more reasonable error structure than a purely one-sided one. Specifically they assume

$$\epsilon_i = \begin{cases} \epsilon_i^* / \sqrt{1-\theta} & \text{if } \epsilon_i^* > 0, \\ \epsilon_i^* / \sqrt{\theta} & \text{if } \epsilon_i^* \leq 0 \end{cases} \quad i = 1, 2, \dots, N \quad (1.3)$$

Where the errors, ϵ_i^* are independent normally distributed random variables with zero means and variance σ^2 for $0 < \theta < 1$; otherwise, ϵ_i^* has either the negative or positive truncated normal distribution, when $\theta = 1$ or $\theta = 0$ respectively.

Their justification for this error specification is that firms are presumed to differ in their “production” of y for a given set of values for the “input” according to random variation in (1.1) their ability to utilize “best practices” technology a source of error that is one-sided ($\epsilon_i \leq 0$), and or (1.2) an input quantity or measurement in y , a symmetric error. The parameter θ is interpreted as the measure of “relative variability” in these two error sources, its values circumscribing the “full” frontier function ($\theta = 1$), the “average” function ($\theta = \frac{1}{2}$), and intermediate cases of some interest.

A primary contribution of this error structure to the literature is that it allows the placement of the fitted function to be estimated along with the usual parameters of interest through the parameter θ . Thus, the criticism levied at the average function by proponents of the frontier [e.g., Aigner and chu (1968)] and criticisms that accompany strict use of the frontier or envelope function as the “appropriate” industry production function [of., Timmer (1971)] are ameliorated by this more accommodating specification. Nevertheless, the interpretation of θ as a measure of the relative variability of error sources is only implicit in the Aigner, Amemiya, Poirier

formulation. A more direct approach is to specifically model the error process implied by the behavioural considerations mentioned above.

II. STOCHASTIC FRONTIER

We now return to the model as given in equation (1.2), but under the error structure

$$\epsilon_i = v_i + u_i \quad i=1, \dots, n \quad (2.1)$$

The error component v_i represents the symmetric disturbance: the $\{v_i\}$ are assumed to be independently and identically distributed as $N(0, \sigma_v^2)$. The error component u_i is assumed to be distributed independently of v_i , and to satisfy $u_i \leq 0$. We will be particularly concerned with the case in which u_i is derived from a $N(0, \sigma_u^2)$ distribution truncated above at zero.

However, other one-sided distributions are tenable, and we will also briefly consider the case in which $-u_i$ has an exponential distribution.

This model collapses to a deterministic frontier model when $\sigma_v^2 = 0$, and it collapses to the Zellner, Kmenta and Dreze (1966) stochastic production function model when $\sigma_u^2 = 0$. Note that $y_i = f(\underline{x}_i; \underline{\beta}) + v_i$, so that the frontier itself is now clearly stochastic.

The economic logic behind this specification is that the production process is subject to two economically distinguishable random disturbances with different characteristics. We believe that there is ample precedent in the literature for such a view, although our interpretation is clearly new. And from a practical standpoint, such a distinction greatly facilitates the estimation and interpretation of frontier. The non-positive disturbance u_i reflects the fact that each firm's output must lie on or below its frontier $[f(\underline{x}_i; \underline{\beta}) + v_i]$. Any such deviation is the result of factors under the firm's control, such as technical and economic inefficiency, the will and effort of the producer and his employees and perhaps such factors as defective and damaged product. But the frontier itself can vary randomly across firms, or over time for the same firm. On this interpretation, the frontier is stochastic, with random disturbance $v_i \geq 0$ being the result of favourable as well as unfavourable external events such as luck, climate, topography, and machine performance. Errors of observation and measurement on y constitute another source of $v_i \geq 0$.

One interesting by product of this approach is that we can estimate the variances of v_i and u_i , so as to get evidence on their relative sizes. Another implication of this approach is that productive efficiency should, in principle, be measured by the ratio

$$y_i / [f(\underline{x}_i; \underline{\beta}) + v_i] \quad (2.2)$$

rather than by the ratio

$$y_i / [f(\underline{x}_i; \underline{\beta})] \quad (2.3)$$

This simply distinguishes productive inefficiency from other sources of disturbance that are beyond the firm's control. For example, the farmer whose crop is decimated by drought or storm is unlucky on our measure (2.2), but inefficient by the usual measure (2.3).

Our discussion of estimation will be simplified somewhat if we consider a linear production function. We therefore write, in obvious matrix form:

$$\underline{Y} = \underline{X}\underline{\beta} + \underline{\epsilon} \quad (2.4)$$

In place of (1.2), where now $\underline{\epsilon} = \underline{v} + \underline{u}$.

III. ESTIMATION OF THE STOCHASTIC FRONTIER MODEL

The distribution function of the sum of a symmetric normal random variable a and a truncated normal random variable was apparently first derived by Weinstein (1964). The derivation of the density function of ϵ is straight forward, so we shall not include it here. The result is

$$f(\epsilon) = f^*\left(\frac{\epsilon}{\sigma}\right) \left[1 - F^*(\epsilon \lambda \sigma^{-1})\right], \quad -\alpha \leq \epsilon \leq +\alpha \quad (3.1)$$

Where $\sigma^2 = \sigma_u^2 + \sigma_v^2$, $\lambda = \sigma_u / \sigma_v$, and $f^*(.)$ and $F^*(.)$ are the standard normal density and distribution functions, respectively. This density is asymmetric around zero, with its mean and variance given by

$$E(\epsilon) = E(u) = -\frac{\sqrt{2}}{\sqrt{\pi}} \sigma_u$$

$$V(\epsilon) = V(u) + V(v) \\ = \left(\frac{\pi - 2}{\pi} \right) \sigma_u^2 + \sigma_v^2 \quad (3.2)$$

as can be easily ascertained from elementary considerations and calculation of the moments of u .

The particular parameterization in (3.1) is convenient because λ is there by interpreted to be an indicator of the relative variability of the two sources of random error that distinguish firms from one another.

$\lambda^2 \rightarrow 0$ implies $\sigma^2 \rightarrow \alpha$ and/or $\sigma_u^2 \rightarrow 0$ i.e., that the symmetric error dominated in the determination of ϵ . Equation (3.1) then becomes the density of a $N(0, \sigma^2)$ random variable, as can be seen by inspection. Similarly, when $\sigma_v^2 \rightarrow 0$, the one-sided error becomes the dominant source of random variation in the model and (2.2) takes on the form of negative half-normal.

The estimation problem is posed by assuming we have available a random sample of n observations and then forming the relevant log-likelihood function,

$$\ln L(\underline{y} / \underline{\beta}; \lambda, \sigma^2) = N \ln \frac{\sqrt{2}}{\sqrt{\pi}} + N \ln \sigma^{-1} + \sum_{i=1}^N \ln \left[1 - F^* \left(\epsilon_i \lambda \sigma^{-1} \right) \right] - \frac{1}{2\sigma^2} \sum_{i=1}^N \epsilon_i^2 \quad (3.3)$$

Note:

1. we prefer to use this interpretation of λ even though σ_u^2 is not the variance of u ;

$\left(\frac{\pi - 2}{\pi} \right) \sigma_u^2$ is. Another useful parameterization is to use σ^2 along with $\infty = \frac{\sigma_u^2}{\sigma^2}$

2. for $\sigma_v^2 = 0$ and thus $\lambda = \infty$, becomes

$$f(t) = \frac{\sqrt{2}}{\sqrt{\pi} \sigma_u} e^{-\frac{1}{2\sigma^2} \epsilon^2} \text{ for } \epsilon \leq 0 \\ = 0, \text{ otherwise}$$

Which is almost exactly the form of the likelihood functions considered by Amemiya (1973). Taking derivatives,

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{N}{2\sigma^4} + \frac{1}{2\sigma^4} \sum_{i=1}^N \left(y_i - \underline{\beta}^1 \underline{x}_i \right)^2 + \frac{\lambda}{2\sigma^3} \sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} \left(y_i - \underline{\beta}^1 \underline{x}_i \right) \quad (3.4)$$

$$\frac{d \ln L}{d \lambda} = -\frac{1}{\sigma} \sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} \left(y_i - \underline{\beta}^1 \underline{x}_i \right) \quad (3.5)$$

$$\frac{d \ln L}{d \underline{\beta}} = \frac{1}{\sigma^2} \sum_{i=1}^N \left(y_i - \underline{\beta}^1 \underline{x}_i \right) \underline{x}_i + \frac{\lambda}{\sigma} \sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} \underline{x}_i \quad (3.6)$$

Where $\underline{x}_i$ is a $(k \times 1)$ vector consisting of elements in the i^{th} row of x and f_i^* and F_i^* are respectively the standard normal density and distribution functions evaluated at $\left(y_i - \underline{\beta}^1 \underline{x}_i \right) \lambda \sigma^{-1}$.

Given (3.2), we have that $\sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} \left(y_i - \underline{\beta}^1 \underline{x}_i \right) = 0$ at the optimum. Inserting this result into

(3.1), the ML estimator for σ^2 is determined through

$$\frac{-N}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^N \left(y_i - \underline{\beta}^1 \underline{x}_i \right)^2 = 0 \quad (3.7)$$

Which yields

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N \left(y_i - \hat{\underline{\beta}}^1 \underline{x}_i \right)^2 \quad (3.8)$$

The basis for the usual ML estimator of residual variance in a regression model. But the determination of $\hat{\beta}$ is not independent of $\hat{\sigma}^2$ from other equations. In any event, this result can be used as a basis for an iterative solution scheme $\hat{\beta}^1$ premultiplied into (3.6) gives

$$\frac{1}{\sigma^2} \sum_{i=1}^N \left(y_i - \hat{\beta}^1 X_i \right) \hat{\beta}^1 X_i + \frac{\lambda}{\sigma} \sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} \hat{\beta}^1 X_i \quad (3.9)$$

Adding to this $-\lambda$ times equation (3.5) and simplifying, we get

$$\frac{1}{\sigma^2} \sum_{i=1}^N \left(y_i - \hat{\beta}^1 X_i \right) \hat{\beta}^1 X_i + \frac{\lambda}{\sigma} \sum_{i=1}^N \frac{f_i^*}{(1 - F_i^*)} y_i \quad (3.10)$$

Which, in conjunction with (3.6) gives a system of $(k+1)$ equations that corresponds very closely to the system of first-order equations encountered in the so-called “Tobit” model.

Regularity conditions that are sufficient for the estimators defined by (3.4), (3.5), and (3.6) to be consistent and asymptotically normal can be found in Amemiya(1973).

Various solution algorithms are available for finding the optimizing values of β , λ , and σ^2 . Most of these (the Fletcher-Powell algorithm, for example) require analytical first - or second-order derivatives in addition to the likelihood function itself for their best performance at reasonable cost in computer time. Since such algorithms are now readily available, we will not devote any space to a discussion of the ML computational problem, except to note that this likelihood function seems to be well-behaved, based on our experience. Second-order derivatives are presented in the appendix to this paper, for that use and as a basis for calculating asymptotic standard errors of the ML estimates.

We note in passing that if estimation of β alone is desired, all but the coefficient in β corresponding to a column of ones in x is estimated un-biasedly and consistently by least squares. Moreover, the components of σ^2 can be extracted (i.e., consistent estimators for them can be found) based on the least squares results by utilizing equation (3.2) for $V(\epsilon)$ in terms of σ_u^2 and σ_v^2 and a similar relationship for a higher order moment of ϵ , since $V(\epsilon)$ and higher order mean-connected moments of ϵ can themselves be consistently estimated from the computed least squares residuals.

Similar comments and derivations would apply under alternative distributional assumptions for u_i . for example, we could choose the simple one parameter exponential distribution for $-u_i$.

$$f(u) = \frac{1}{\phi} \exp(u / \phi), u \leq 0 \quad (3.11)$$

Where $\Phi \geq 0$ is the mean of $-u_i$ (the variance is ϕ^2). A little algebra reveals that the distribution of

$\epsilon_i = v_i + u_i$ is given by the density

$$f(\epsilon) = \frac{1}{\phi} \left[1 - F^* \left(\frac{\epsilon}{\sigma_v} + \frac{\sigma_v}{\phi} \right) \right] \exp \left[\frac{\epsilon}{\phi} + \frac{\sigma_v^2}{2\phi^2} \right] \quad (3.12)$$

Where again $F^*(.)$ represents the cumulative distribution function of the standard normal distribution. The likelihood function for the model follows immediately.

IV. STOCHASTIC PRODUCTION FRONTIERS

Stochastic production frontiers were initially developed for estimating technical efficiency rather than capacity and capacity utilization. However, the technique also can be applied to capacity estimation through modification of the inputs incorporated in production (or distance) function. A potential advantage of the stochastic production frontier approach over DEA is that random variations in catch can be accommodated, so that the measure is more consistent with the potential harvest under “normal” working conditions. A disadvantage of the technique is that, although it can model multiple output technologies, doing so is somewhat more complicated requires stochastic multiple output distance functions, and raises problems for output that take zero values.

V. THE UNDERLYING THEORY

A production function defines the technological relationship between the level of inputs and the resulting level of outputs. If estimated econometrically from data on observed outputs and input usage, it indicates the average level of outputs that can be produced from a given level of inputs. A number of studies have estimated the relative contributions of the factors of production through estimating production functions at either the individual boat level or total fishery level. These include Cobb-Douglas production functions, CES production functions (Canbell and Lindner, 1990) and trans-log production functions (Squires, 1987 ; Pascoe and Robinson, 1998)

An implicit assumption of production functions is that all firms are producing in a technically efficient manner, and the representative (average) firm therefore defines the frontier variations from the frontier are thus assumed to be random, and are likely to be associated with mis-or un-measured production factors. In contrast, estimation of the production frontier assumes that the boundary of the production function is defined by “best practice” firms. It therefore indicates the maximum potential output for a given set of inputs along a ray from the origin point. Some white noise is accommodated, since the estimation procedures are stochastic, but an additional one-sided error represent any other reason firms would be away from (within) the boundary. Observations with in the frontier are deemed “inefficient”. So from an estimated production frontier it is possible to measure the relative efficiency of certain groups or a set of practices from the relationship between observed production and some ideal or potential production (Greene, 1993).

A general stochastic production frontier model can be given by

$$\ln q_j = f(\ln x) + v_j - u_j \quad (5.1)$$

Where q_j is the output produced by firm j , x is a vector of factor inputs, v_j is the stochastic (white noise) error term and u_j is a one-sided error representing the technical inefficiency of firm j . both v_j and u_j are assumed to be independently and identically distributed (IID) with variance σ_v^2 and σ_u^2 respectively.

Given that the production of each firm j can be estimated as:

$$\ln \hat{q}_j = f(\ln x) - u_j \quad (5.2)$$

While the efficient level of production (i.e., no inefficiency) is defined as:

$$\ln q_j = f(\ln x) \quad (5.3)$$

Then technical efficiency (TE) can be given by

$$\ln TE_j = \ln \hat{q}_j - \ln q_j^* = -u_j \quad (5.4)$$

Hence, $TE_j = e^{-u_j}$, and is constrained to be between zero and one in value. If u_j equals zero, then TE_j equals one, and production is said to be technically efficient. Technical efficiency of the j^{th} firm is therefore a relative measure of its output as a proportion of the corresponding frontier output. A firm is technically efficient if its output level is on the frontier, which implies that q/q^* equals one in value.

While the techniques have been developed primarily to estimate efficiency, they can be readily modified to represent capacity utilization. In estimation the full utilization production frontier, a distinction must be made between inputs comprising the capacity base (usually capital inputs), and variable inputs (usually days, or variable effort). If capacity is defined only in terms of capital inputs, the implied variation in output, and thus variable effort, from its full utilization level is sometimes termed an indicator of capital utilization. If variable inputs are assumed to be approximated by the number of hours or days fished (i.e., normal units of effort), estimating the potential output producible from the capacity base with variable inputs “unconstrained” implies removing this variable from the estimation of the frontier. The resulting production frontier is thus defined only in terms of the fixed factors of production or k . In particular, it will be supported by observations for the boats that have the greatest catch per unit of fixed input (which generally corresponds to the boats that employ the greatest level of nominal effort for a particular level of k). The resulting measure of technical efficient capacity utilization (TECU); accommodating both the impacts of technical inefficiency and deviations from full utilization of the capacity base. That is, it represents the ratio of the potential capacity output that could be achieved if all fixed inputs were being utilized efficiently and fully to observed output.

VI. FUNCTION FORMS FOR THE PRODUCTION FUNCTION

Estimation of the SPE requires a particular functional form of the production function to be imposed. A range of functional forms for the production function frontier are available, with the most frequently used being a trans-log function, which is a second order (all cross-terms included) log-linear form. This is a relatively flexible functional form, as it does not impose assumptions about constant elasticity of production nor elasticities of

substitution between inputs. It thus allows the data to indicate the actual curvature of the function, rather than imposing a priori assumptions. In general terms, this can be expressed as

$$\ln Q_{jt} = \beta_0 + \sum_i \beta_i \ln X_{j,i,t} + \frac{1}{2} \sum_i \sum_k \beta_{i,k} \ln x_{j,i,t} \ln x_{j,k,t} - u_{j,t} + v_{j,t} \quad (6.1)$$

Where Q_{jt} is the output of the vessel j in period t and $X_{j,i,t}$ and $X_{j,k,t}$ are the variable and fixed vessel inputs (i, k) to the production process. As noted above, the error term is separated into two components, where $v_{j,t}$ is the stochastic error term and $U_{j,t}$ is an estimate of technical inefficiency.

Alternative production functions include the Cobb-Douglas and CES (Constant Elasticity of Substitution) production functions. The Cobb-Douglas production function is given by

$$\ln Q_{ji} = \beta_0 + \sum_i \beta_i \ln X_{j,i,t} - u_{j,t} + v_{j,t} \quad (6.2)$$

As can be seen, the Cobb-Douglas is a special case of the trans-log production function where all $b_{i,k}=0$. The production function imposes more stringent assumptions on the data than the trans-log, because the elasticity of substitution has constant value of 1 (i.e. the functional form assumption imposes a fixed degree of substitutability on all inputs). And the elasticity of production is constant for all inputs (i.e., a 1 percent change in input level will produce the same percentage change in output, irrespective of any other arguments of the function).

The CES production function is given by.

$$Q_{j,t} = \gamma \left[\delta x_{1,j,t} + (1-\delta) x_{2,j,t} \right]^{-1/\theta} - u_{j,t} + v_{j,t} \quad (6.3)$$

Where q is the substitution parameter related to the elasticity of substitution (i.e., $q=(1/s)-1$ where 's' is the elasticity of substitution) and d is the distribution parameter. The CES production function is limited to two variables, and is not possible to estimate in the form given in (3.7) in maximum likelihood estimation (MLE) (making it unsuitable for use as the basis of a production frontier). However, a Taylor series expansion of the function yields a functional form of the model that can be estimated, given as:

$$\ln \left(\frac{Q_{j,t}}{x_{2,j,t}} \right) = \ln \gamma + (u-1) \ln x_{2,j,t} + v \delta \ln \left(\frac{x_{1,j,t}}{x_{2,j,t}} \right) - \frac{1}{2} u \theta \delta (1-\delta) \left[\ln \left(\frac{x_{1,j,t}}{x_{2,j,t}} \right) \right]^2 - u_{j,t} + v_{j,t} \quad (6.4)$$

The model can be estimated as a standard or frontier production function, and the parameter values derived through manipulation of the regression co-efficient. The functional form in (3.8) can be shown to be a special case of the trans-log function.

Given that both the Cobb-Douglas and CES production functions are special cases of the trans-log, ideally the trans-log should be estimated first and the restrictions outlined above, tested. However, the large number of variables required in the process of estimating the trans-log may cause problems if a sufficient data series is not available, resulting in degree of freedom problems. In such a case, more restrictive assumptions must be imposed.

VII. MATHEMATICAL PROGRAMMING FOR THE MEASUREMENT OF TECHNICAL EFFICIENCY

To assure the technical efficiency one may use a priority specified neo-classical production function such as the Cobb – Douglas Production function.

$$u = \lambda x_1^\alpha x_2^\beta$$

Where u = output ; x_1, x_2 = inputs.

Let there be n production units whose output lie on or below the Frontier Production function:

$$\hat{u} = \hat{\lambda} x_1^{\hat{\alpha}} x_2^{\hat{\beta}} \quad (7.1)$$

Let $(u_i, x_{1i}, x_{2i}), i=1,2,\dots,n$

Be the output – input vector of i^{th} production unit had the i^{th} production unit been technically efficient its output-input vector is given by.

$$(\hat{u}_i, x_{1i}, x_{2i}), i=1,2,\dots,n$$

Thus, we have,

$$\begin{aligned} \hat{u}_i &\geq u_i, i=1,2,\dots,n \\ \Rightarrow \ln \hat{u}_i &\geq \ln u_i, i=1,2,\dots,n \end{aligned}$$

$$\ln \hat{\lambda} + \hat{\alpha} \ln x_{1i} + \hat{\beta} \ln x_{2i} \geq \ln u_i, i=1, 2, \dots, n$$

We hypothesise that the n production units compete with each other to achieve technical efficiency. Thus, the objective is to minimize,

$$\sum_{i=1}^n (\ln \hat{u}_i - \ln u_i) \quad (7.2)$$

Subject to

$$\begin{aligned} \ln \hat{\lambda} + \hat{\alpha} \ln x_{1i} + \hat{\beta} \ln x_{2i} &\geq u_i \\ \ln \hat{\lambda} &\geq 0, \hat{\alpha} \geq 0, \hat{\beta} \geq 0 \end{aligned} \quad (7.3)$$

$i=1, 2, \dots, N$

Minimizing (7.2) is same as minimizing

$$\ln \hat{\lambda} + \hat{\alpha} \ln \bar{x}_1 + \hat{\beta} \ln \bar{x}_2 \quad (7.4)$$

The expression (7.4) is the objective function. Thus, to measure unit wise technical efficiency one may solve the following linear programming problem.

$$\text{Min } \hat{\alpha}_0 + \hat{\alpha} \bar{x}_1 + \hat{\beta} \bar{x}_2 \quad (7.5)$$

$$\text{Subject to } \hat{\alpha}_0 + \hat{\alpha} \bar{x}_{1i} + \hat{\beta} \bar{x}_{2i} \geq u_i \quad (7.6)$$

$i=1, 2, \dots, n.$

$$\hat{\alpha}_0, \hat{\alpha}, \hat{\beta} \geq 0 \quad (7.7)$$

Where $\hat{\alpha}_0 = \ln \hat{\lambda}$ $x_1 = \ln x_1$

$$x_2 = \ln x_2 \quad u = \ln u$$

For the above linear programming problem if optimal solution exists all artificial variables have to assume zero values. Thus, for i^{th} production unit one has,

$$\hat{\alpha}_0 + \hat{\alpha} x_{1i} + \hat{\beta} x_{2i} - S_i = u_i$$

Where S_i is the surplus variable of i^{th} constraint.

$$S_i \geq 0 \quad i=1, 2, \dots, n$$

$$\begin{aligned} -S_i &= u_i - (\hat{\alpha}_0 + \hat{\alpha} x_{1i} + \hat{\beta} x_{2i}) \\ &= \ln u_i - \ln \hat{u}_i \\ &= \ln u_i / \hat{u}_i \\ \text{TE}(i) &= \frac{u_i}{\hat{u}_i} = e^{-S_i} \end{aligned} \quad (7.8)$$

The expression (7.8) gives the technical efficiency of i^{th} production unit. Thus, we have $0 \leq \text{TE}(i) \leq 1$, $i=1, 2, \dots, n$.

This method of measuring technical efficiency is due to TIMMER. The Timmer's method solves a single linear programming problem to compute production unit wise technical efficiency. A different method to measure technical efficiency of i^{th} production unit is, to

$$\begin{aligned} \text{Maximize } &\hat{\alpha}_0 + \hat{\alpha} x_{1i} + \hat{\beta} x_{2i} \\ \text{Subject to } &\hat{\alpha}_0 + \hat{\alpha} x_{1i} + \hat{\beta} x_{2i} \geq u_i \end{aligned}$$

$i=1, 2, \dots, n.$

If there are n production units, this method requires to solve n linear programming problems.

References

- [1]. Afraita, S.N (1972) , "Efficiency Estimation of Production Functions," international economic review, vol. 13(October), pp.568-98.
- [2]. Aigner, D.J., and S.F.Chu (1968)" On estimating the industry production function, " American Economic Review, Vol. 58, pp.826-839.
- [3]. Aigner ,D.J., T.Amemiya and D.J.Poirier (1976),"On the Estimation of Production Frontiers, "International Economic Review.
- [4]. Aigner,D., C.A.K.Lovell and P.Schmidt,(1977), Formulation and Estimation of Stochastic Frontier Production Function models, Journal of Econometrics, vol.6,pp.21-37.

- [5]. Dugger, R., (1974), An application of Bouded Non parametric estimating functions to the analysis of bank cost and production functions, unpublished Ph.D. dissertation (University of North Carolina, Chapel Hill, NC).
- [6]. Richmod, J.(1974), "Estimating the Efficiency of Production , International Economic Review, vol.15,pp.515-21.
- [7]. Timmer, C.Peter (1971), "using a Probalistic Frontier Production Function to Measure Technical Efficiency, "Journal of Political Economy, vol.79,pp776-794.
- [9]. Zellner, A., Kmenta, J., and J.Dreze(1966), "Specification and Estimation of Cobb-Douglas Production Functions,"Econometrica, vol.34,pp.784-95.