



UPPER VERTEX COVERING NUMBER OF A GRAPH

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Abstract: In this paper we consider upper vertex covering sets of a graph and its upper vertex covering number. We prove that upper vertex covering number of a graph does not increase when a vertex is removed from the graph. We also prove necessary and sufficient condition in which this number does not changed. We also consider well covered graphs and prove some interesting results. We further prove that if G is approximately well dominated graph then G is either well covered or G is approximately well covered.

Keywords: Γ_{cr} – sets, upper vertex covering number, vertex covering number, α_0 – set, well covered graphs, well dominated graph, approximately well dominated graph.

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I. Introduction

In this paper we consider vertex covering set in graphs. We will define Γ_{cr} sets which are infect minimal vertex covering sets with maximum cardinality. We will find conditions under which the upper vertex covering number of a graph decreases or remains same. Before that we will prove that this number never increases when a vertex is removed. Well cover graphs have been studied in past [2]. However we consider these graphs here for different reason.

II. Preliminaries

Definition: 2.1 [4]

A subset S of $V(G)$ is said to be a vertex covering set if for every edge of the graph at least one end vertex of it is in S .

Definition: 2.2 [4]

A vertex covering set S is said to be a minimal vertex covering set if $S - v$ is not a vertex covering set for every v in S .

Definition: 2.3

A vertex covering set S with minimum cardinality is called a minimum vertex covering set and is denoted as α_0 – set.

The cardinality of a minimum vertex covering set of a graph G is called the vertex covering number of G and is denoted as $\alpha_0(G)$.

Definition: 2.4

A minimal vertex covering set with maximum cardinality is called Γ_{cr} – set.

The cardinality of Γ_{cr} – set is called the upper vertex covering number of the graph G and is denoted as $\Gamma_{cr}(G)$.

Obviously, $\alpha_0(G) \leq \Gamma_{cr}(G)$.

In this paper we will consider only simple graphs with finite vertex set.

III. Main Result

First we mentioned the following notations [3]

$$V_{cr}^+ = \{v \in V(G): \alpha_0(G - v) > \alpha_0(G)\}$$

$$V_{cr}^- = \{v \in V(G): \alpha_0(G - v) < \alpha_0(G)\}$$

$$V_{cr}^0 = \{v \in V(G): \alpha_0(G - v) = \alpha_0(G)\}$$

The above sets are mutually disjoint and their union = $V(G)$.

First, we prove that for any graph G , V_{cr}^+ is empty.

Theorem: 3.1 Let G be a graph and $v \in V(G)$ then $\alpha_0(G - v) \leq \alpha_0(G)$. [3]

Theorem: 3.2 Let G be a graph, v be a vertex of G such that $v \in V_{cr}^-$ then $\alpha_0(G - v) = \alpha_0(G) - 1$. [3]

Now we prove that the upper vertex covering number does not increase when a vertex is removed from the graph.

Theorem: 3.3 Let G be a graph and $v \in V(G)$. Then $\Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Proof: Let S be a Γ_{cr} set of $G - v$. If all the neighbours of v are in S then S is a minimal vertex covering set of G and therefore $|S| \leq \Gamma_{cr}(G)$ and thus $\Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

If some neighbor of v is not in S then $S \cup \{v\}$ is a minimal vertex covering set of graph G .

Therefore, $|S| < |S \cup \{v\}| \leq \Gamma_{cr}(G)$. That is $\Gamma_{cr}(G - v) < \Gamma_{cr}(G)$

We define the following symbols.

$$W_{cr}^+ = \{v \in V(G) : \Gamma_{cr}(G - v) > \Gamma_{cr}(G)\}$$

$$W_{cr}^- = \{v \in V(G) : \Gamma_{cr}(G - v) < \Gamma_{cr}(G)\}$$

$$W_{cr}^0 = \{v \in V(G) : \Gamma_{cr}(G - v) = \Gamma_{cr}(G)\}$$

We now prove the following theorem.

Theorem: 3.4 Let G be a graph and $v \in V(G)$. Then $v \in W_{cr}^0$ if and only if there is a Γ_{cr} - set S of G not containing v such that S is also Γ_{cr} - set of $G - v$.

Proof: Suppose that $v \in W_{cr}^0$.

Let S_1 be any Γ_{cr} - set of $G - v$. If some neighbor of v is not in S_1 then $S = S_1 \cup \{v\}$ is a minimal vertex covering set of G and hence $|S_1| < |S| \leq \Gamma_{cr}(G)$.

That is $\Gamma_{cr}(G - v) < \Gamma_{cr}(G)$. Which implies that $v \in W_{cr}^-$. Which is not true. Thus, all neighbors of v must in S_1 . Let $S = S_1$ then as proved in previous theorem S is a minimal vertex covering set of G . If S is not a Γ_{cr} - set of G then $|S| < \Gamma_{cr}(G)$.

That is $\Gamma_{cr}(G - v) < \Gamma_{cr}(G)$. Which is a contradiction.

Hence, S is a Γ_{cr} - set of G . Also, thus S is the required Γ_{cr} - set.

Conversely, suppose S is a Γ_{cr} - set of G not containing v such that S is also a Γ_{cr} - set of $G - v$ then $\Gamma_{cr}(G) = |S| = \Gamma_{cr}(G - v)$. Thus, $v \in W_{cr}^0$

Corollary: 3.5 Let G be a graph and $v \in V(G)$. Then $v \in W_{cr}^-$ if and only if whenever S is a Γ_{cr} - set of G not containing v then S is not a Γ_{cr} - set of $G - v$

Example: 3.6

Consider the wheel graph with six vertices.

Then $\Gamma_{cr}(G) = 3$. Let $v = 0$ then $G - 0 = C_5$ and $\Gamma_{cr}(G - v) = 3$. Thus, $v \in W_{cr}^0$.

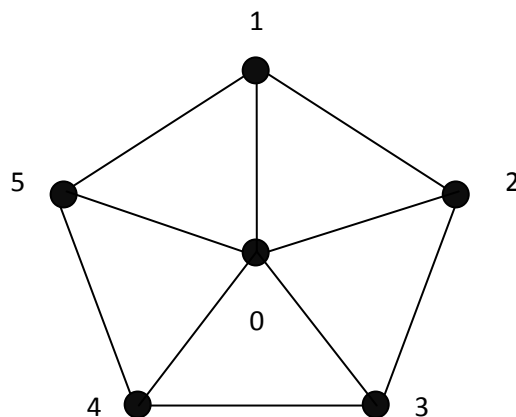


Figure: 1

In fact $S = \{1,3,4\}$ is a Γ_{cr} - set of G not containing 0 such that S is a Γ_{cr} - set of $G - v$.

IV. Well Covered Graphs

Definition: 4.1 [4]

A graph G is said to be a well covered if any two minimal vertex covering sets have the same cardinality.

Equivalently a graph G is well covered if $\alpha_0(G) = \Gamma_{cr}(G)$.

For example C_4 and P_4 are well covered graphs.

Definition: 4.2

A graph G is said to be an approximately well covered graph if $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

For example Peterson graph is an approximately well covered graph.

Definition: 4.3[1]

A graph G is said to a well dominated graph if any two minimal dominating sets have the same cardinality.

Equivalently a graph G is well dominated if $\gamma(G) = \Gamma(G)$.

Definition: 4.4[1]

A graph G is said to be an approximately well dominated graph if $\gamma(G) = \Gamma(G) - 1$.

Theorem: 4.5 Let G be a well covered graph and $v \in V(G)$. Then

- 1) $G - v$ is well covered or $v \in W_{cr}^0$.
- 2) If $v \in V_{cr}^0$ then $v \in W_{cr}^0$ and $G - v$ is well covered.
- 3) If $v \in V_{cr}^-$ then either $G - v$ is well covered and $\Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$ or $v \in W_{cr}^0$.

Proof: 1) $\alpha_0(G - v) \leq \alpha_0(G) \leq \Gamma_{cr}(G)$.

Also $\alpha_0(G - v) \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$ Hence, if $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ then $G - v$ is well covered or if $\Gamma_{cr}(G - v) = \Gamma_{cr}(G)$ then $v \in W_{cr}^0$.

2) In this case $\alpha_0(G - v) = \alpha_0(G) \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Therefore, $\alpha_0(G - v) = \alpha_0(G) = \Gamma_{cr}(G - v) = \Gamma_{cr}(G)$.

Thus, $G - v$ is well covered and $v \in W_{cr}^0$.

3) $\alpha_0(G - v) = \alpha_0(G) - 1 \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Therefore, $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ or $\Gamma_{cr}(G - v) = \alpha_0(G) = \Gamma_{cr}(G)$.

Thus, either $G - v$ is well covered and $\Gamma_{cr}(G - v) = \Gamma_{cr}(G) - 1$ or $v \in W_{cr}^0$.

Theorem: 4.6 Let G be an approximately well covered graph and $v \in V(G)$. Then

- 1) If $v \in V_{cr}^0$ then either $G - v$ is well covered or approximately well covered.
- 2) If $v \in V_{cr}^-$ then either $G - v$ is well covered or approximately well covered or $v \in W_{cr}^0$.

Proof: 1) $\alpha_0(G - v) = \alpha_0(G) \leq \Gamma_{cr}(G)$. Also $\alpha_0(G - v) \leq \Gamma_{cr}(G - v) \leq \Gamma_{cr}(G)$.

Thus, if $v \in V_{cr}^0$ then $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ and in this case $G - v$ is well covered or if $\Gamma_{cr}(G - v) = \Gamma_{cr}(G)$.

Then $\alpha_0(G - v) = \alpha_0(G) = \Gamma_{cr}(G) - 1 = \Gamma_{cr}(G - v) - 1$ and hence $G - v$ is approximately well covered.

2) If $v \in V_{cr}^-$ then if $\Gamma_{cr}(G - v) = \alpha_0(G - v)$ then $G - v$ is well covered. If $\Gamma_{cr}(G - v) = \alpha_0(G)$ then $\Gamma_{cr}(G - v) = \alpha_0(G) = \Gamma_{cr}(G) - 1$. Then $G - v$ is approximately well covered. If $\Gamma_{cr}(G - v) = \Gamma_{cr}(G)$ then $v \in W_{cr}^0$.

Theorem: 4.7 If graph G is approximately well dominated then either G is well covered or G is approximately well covered.

Proof: Since G is approximately well dominated, $\gamma(G) = \Gamma(G) - 1$. Now every maximal independent set is a minimal dominating set. Therefore cardinality of every maximal independent set is equal to $\Gamma(G) - 1$ or $\Gamma(G)$. Therefore $i(G) = \Gamma(G)$ or $i(G) = \Gamma(G) - 1$. Now $i(G) \leq \beta_0(G) \leq \Gamma(G)$ (Because a maximum independent set is a minimal dominating set).

Case:I $i(G) = \Gamma(G)$.

Then from the above inequality $\beta_0(G) = \Gamma(G) = i(G)$. Therefore $n - \beta_0(G) \leq n - i(G)$. Now $\alpha_0(G) + \beta_0(G) = n$ and $i(G) + \Gamma_{cr}(G) = n$. Thus, $\alpha_0(G) = \Gamma_{cr}(G)$. Therefore the graph is well covered.

Case:II $i(G) = \Gamma(G) - 1$.

Now again, $i(G) \leq \beta_0(G) \leq \Gamma(G)$. If $\beta_0(G) = \Gamma(G) - 1$. Then $\beta_0(G) = i(G)$. Therefore by the argument in case I G is well covered.

Suppose $\beta_0(G) = \Gamma(G)$ then $i(G) = \beta_0(G) - 1$. Therefore $n - i(G) \leq (n - \beta_0(G)) + 1$. Therefore $\Gamma_{cr}(G) = \alpha_0(G) + 1$. Therefore, $\alpha_0(G) = \Gamma_{cr}(G) - 1$. Hence, $\alpha_0(G) = \Gamma_{cr}(G) - 1$.

V. References

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