



FIXED POINT THEOREMS FOR WIDELY GENERALIZED MAPPINGS IN HILBERT SPACES AND APPLICATIONS

Mamta Patel¹, Sanjay Sharma²

¹Department of Applied Mathematics,

Chhatrapati Shivaji Institute of Technology, Durg (C.G), INDIA

²Department of Applied Mathematics

Bhilai Institute of Technology, Durg (C.G) INDIA

Abstract: Recently Kawasaki and Takahashi defined a broad class of non-linear mappings, widely generalized hybrid in a Hilbert space which contains generalized hybrid mappings. Here, in this paper we introduce a broad class of non-linear mappings in a Hilbert space. Then we prove the fixed point theorems for the class of such mappings.

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I. Introduction

Let H be a Hilbert space and C be a non empty subset of H . A mapping $T: C \rightarrow H$ is said to be a *nonexpansive* [10], *nonspreading* [13], and *hybrid* [9] if

$$Tx - Ty \leq x - y$$

$$2Tx - Ty^2 \leq Tx - y^2 + Ty - x^2$$

$$\text{and } 3Tx - Ty^2 \leq x - y^2 + Tx - y^2 + Ty - x^2$$

for all $x, y \in C$, respectively; see also [12], [11], and [7]. These mappings are independent and they are deduced from a *firmly nonexpansive* mapping in a Hilbert space; see [9]. A mapping $F: C \rightarrow H$ is said to be firmly nonexpansive if

$$Fx - Fy^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$.

A mapping T from C into H is said to be *generalized hybrid* if there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha Tx - Ty^2 + (1 - \alpha)x - Ty^2 \leq \beta Tx - y^2 + (1 - \beta)x - y^2$$

for any $x, y \in C$.

A mapping T from C into H is said to be *widely generalized hybrid* if there exists $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$ such that

$$\alpha Tx - Ty^2 + \beta x - Ty^2 + \gamma Tx - y^2 + \delta x - y^2 + \max\{\epsilon x - Tx^2, \zeta y - Ty^2\} \leq 0$$

for any $x, y \in C$.

A mapping T from C into H is said to be *widely more generalized hybrid* if there exists $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\alpha Tx - Ty^2 + \beta x - Ty^2 + \gamma Tx - y^2 + \delta x - y^2 + \epsilon x - Tx^2 + \zeta y - Ty^2 + \eta(x - Tx) - (y - Ty)^2 \leq 0$$

II. Preliminaries

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. Let A be a non empty subset of H . We denote by $\overline{\text{co}} A$ the closure of the convex hull of A . In a Hilbert space, it is known that

$$(2.1) \quad \alpha x + (1 - \alpha)y^2 = \alpha x^2 + (1 - \alpha)y^2 - \alpha(1 - \alpha)x - y^2$$

For all $x, y \in H$ and $\alpha \in \mathbb{R}$; Furthermore, in a Hilbert space, we have that

$$(2.2) \quad 2 \langle x - y, z - w \rangle = x - w^2 + y - z^2 - x - z^2 - y - w^2$$

For all $x, y, z, w \in H$. Let C be a non empty subset of H and let T be a mapping from C to H . We denote by $F(T)$ the set of fixed points of T .

Lemma2.1. Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞ . Then there exists a unique point $z_0 \in \overline{\text{co}}\{x_n | n \in N\}$ such that $\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H$.

III. Fixed Points Theorems

Let H be a real Hilbert space and let C be a non empty subset of H . A mapping T from C into H is called **Symmetric generalized hybrid** if there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$(3.1) \quad \alpha Tx - Ty^2 + \beta(x - Ty^2 + Tx - y^2 + \gamma x - y^2 + \delta(x - Tx^2 + y - Ty^2)) \leq 0$$

for all $x, y \in C$. Such a mapping T is also called $(\alpha, \beta, \gamma, \delta)$ – **symmetric generalized hybrid**.

A mapping T from C into H is said to be **widely generalized hybrid** if there exists $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$(3.2) \quad \alpha Tx - Ty^2 + \beta x - Ty^2 + \gamma Tx - y^2 + \delta x - y^2 + \max\{\varepsilon x - Tx^2, \zeta y - Ty^2\} \leq 0$$

for all $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ – **widely generalized hybrid**.

Theorem3.1. Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$ – **symmetric generalized hybrid mapping** from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\beta + \gamma + \delta > 0$ and (3) $\delta \geq 0$ hold. Then T has a fixed if and there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is a unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

Proof:

Suppose that T has a fixed point z . Then $\{T^n z : n = 0, 1, \dots\} = \{z\}$ and hence $\{T^n z : n = 0, 1, \dots\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. Since T is a $(\alpha, \beta, \gamma, \delta)$ – **symmetric generalized hybrid mapping** of C into itself, we have that

$$\alpha Tx - T^n z^2 + \beta(x - T^{n+1} z^2 + Tx - T^n z^2) + \gamma x - T^n z^2 + \delta(x - Tx^2 + T^n z - T^{n+1} z^2) \leq 0$$

For all $n \in N \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ on both sides of the inequality. Since $\mu_n Tx - T^n z^2 = \mu_n Tx - T^{n+1} z^2$ and

$$\mu_n x - T^{n+1} z^2 = \mu_n x - T^n z^2, \text{ we have that}$$

$$(\alpha + \beta)\mu_n Tx - T^n z^2 + (\beta + \gamma)\mu_n x - T^n z^2 + \delta(x - Tx^2 + \mu_n T^n z - T^{n+1} z^2) \leq 0$$

Furthermore, since

$$\mu_n x - T^n z^2 = x - Tx^2 + 2\mu_n \langle x - Tx, Tx - T^n z \rangle + \mu_n Tx - T^n z^2,$$

We have that

$$(\beta + \gamma + \delta)x - Tx^2 + 2(\beta + \gamma)\mu_n \langle x - Tx, Tx - T^n z \rangle + (\alpha + 2\beta + \gamma)\mu_n Tx - T^n z^2 + \delta\mu_n T^n z - T^{n+1} z^2 \leq 0$$

From (1) $\alpha + 2\beta + \gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\beta + \gamma + \delta)x - Tx^2 + 2(\beta + \gamma)\mu_n \langle x - Tx, Tx - T^n z \rangle \leq 0 \dots \dots \dots (1)$$

Since there exists $p \in H$ from **lemma2.1** such that

$$\mu_n \langle y, T^n z \rangle = \langle y, p \rangle$$

for all $y \in H$, we have from equation 1 that

$$(\beta + \gamma + \delta)x - Tx^2 + 2(\beta + \gamma)\mu_n \langle x - Tx, Tx - p \rangle \leq 0 \dots \dots \dots (2)$$

Since C is closed and convex, we have that

$$p \in \overline{\text{co}}\{T^n x | n \in N\} \subset C$$

Putting $x = p$, we obtain from equation (2) that

$$(\beta + \gamma + \delta)p - Tp^2 \leq 0$$

We have from equation (2)

$$(\beta + \gamma + \delta) > 0 \text{ that } p - Tp^2 \leq 0. \text{ This implies that } p \text{ is a fixed point in } T.$$

Next suppose that $(\alpha + 2\beta + \gamma) > 0$. Let p_1 and p_2 be fixed points of T . Then we have that

$$\alpha Tp_1 - Tp_2^2 + \beta(p_1 - Tp_2^2 + Tp_1 - p_2^2) + \gamma p_1 - p_2^2 + \delta(p_1 - Tp_1^2 + p_2 - Tp_2^2) \leq 0$$

And hence $(\alpha + 2\beta + \gamma)p - Tp^2 \leq 0$.

We have from $(\alpha + 2\beta + \gamma) > 0$ that $p_1 = p_2$.

Therefore a fixed point theorem is unique. This completes the proof.

Theorem 3.2. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\beta + \gamma + \delta > 0$ and (3) $\delta \geq 0$ hold. Then T has a fixed point. In particular, a fixed point of T is a unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

Using Theorem 3.1 we prove the following fixed point theorem. Before proving it, we introduce a more broad class of non-linear mappings which contains the class of symmetric generalized hybrid mappings. A mapping T from C to H is called *Symmetric More generalized hybrid* if there exists

$(\alpha, \beta, \gamma, \delta, \zeta) \in \mathbb{R}$ such that

$$\alpha Tx - Ty^2 + \beta(x - Ty^2 + Tx - y^2) + \gamma x - y^2 + \delta(x - Tx^2 + y - Ty^2) + \zeta x - y - (Tx - Ty^2) \leq 0$$

for all $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid.

Theorem 3.3 Let H be a Hilbert space, let C be a non empty closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping from C into H satisfies the following conditions (1) and (2).

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma + \varepsilon > 0 \text{ and } \alpha + \beta \geq 0$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta + \zeta > 0 \text{ and } \alpha + \gamma \geq 0$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z: n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is a unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

Proof:

Suppose that T has a fixed point z . Then $\{T^n z: n = 0, 1, \dots\} = \{z\}$ and hence $\{T^n z: n = 0, 1, \dots\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z: n = 0, 1, \dots\}$ is bounded. Since T is a $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely generalized hybrid mapping of, we have that

$$\alpha Tx - T^{n+1}z^2 + \beta(x - T^{n+1}z^2 + \gamma Tx - T^n z^2) + \delta x - T^n z^2 + \max\{\varepsilon x - Tx^2, \zeta T^n z - T^{n+1}z^2\} \leq 0$$

for any $x, y \in C$. In the case of $\alpha + \gamma + \varepsilon > 0$

from

$$\varepsilon x - Tx^2 \leq \max\{\varepsilon x - Tx^2, \zeta T^n z - T^{n+1}z^2\}$$

We obtain that

$$\alpha Tx - T^{n+1}z^2 + \beta(x - T^{n+1}z^2 + \gamma Tx - T^n z^2) + \delta x - T^n z^2 + \varepsilon x - Tx^2 \leq 0$$

i.e it is an $(\alpha, \beta, \gamma, \delta, 0, 0)$ -widely more generalized hybrid mapping

$\{T^n z\}$ is bounded, we can apply a Banach Limit μ to both sides of inequality.

Since $\mu_n Tx - T^n z^2 = \mu_n Tx - T^{n+1}z^2$ and

$$\mu_n x - T^{n+1}z^2 = \mu_n x - T^n z^2$$

So we get

$$\alpha \mu_n Tx - T^n z^2 + \beta \mu_n x - T^n z^2 + \gamma \mu_n Tx - T^n z^2 + \delta \mu_n x - T^n z^2 + \varepsilon x - Tx^2 \leq 0$$

$$(\alpha + \gamma) \mu_n Tx - T^n z^2 + (\beta + \delta) \mu_n x - T^n z^2 + \varepsilon x - Tx^2 \leq 0$$

Furthermore, since

$$\mu_n Tx - T^n z^2 = Tx - x^2 + 2\mu_n \langle Tx - x, x - T^n z \rangle + \mu_n x - T^n z^2$$

We have

$$(\alpha + \gamma)[Tx - x^2 + 2\mu_n \langle Tx - x, x - T^n z \rangle + \mu_n x - T^n z^2] + (\beta + \delta) \mu_n x - T^n z^2 + \varepsilon x - Tx^2 \leq 0$$

$$(\alpha + \gamma + \varepsilon)Tx - x^2 + 2\mu_n(\alpha + \gamma) \langle Tx - x, x - T^n z \rangle + (\alpha + \beta + \gamma + \delta) \mu_n x - T^n z^2 \leq 0$$

We have from equation (1). $(\alpha + \beta + \gamma + \delta) \geq 0$

$$(\alpha + \gamma + \varepsilon)Tx - x^2 + 2\mu_n(\alpha + \gamma) \langle Tx - x, x - T^n z \rangle \leq 0$$

Since $\exists p \in H$ from lemma 2.1 such that

$$\mu_n \langle y, T^n z \rangle = \langle y, p \rangle \text{ for all } y \in H$$

We have

$$(\alpha + \gamma + \varepsilon)Tx - x^2 + 2\mu_n(\alpha + \gamma) \langle Tx - x, x - p \rangle \leq 0$$

Since C is closed and convex we have that

$$p \in \overline{\text{co}}\{T^n x | n \in \mathbb{N}\} \subset C$$

Putting $p = p$, we obtain

$$(\alpha + \gamma + \varepsilon)p - Tp^2 \leq 0$$

We have from equation (1) $\alpha + \gamma + \varepsilon > 0$

$(\alpha + \gamma + \varepsilon) > 0$ that $p - Tp^2 \leq 0$.

This implies that p is a fixed point in T .

Next suppose that from (2) $(\alpha + \beta + \zeta) > 0$. Let p_1 and p_2 be fixed points of T . Then we have that

$$\alpha Tp_1 - Tp_2^2 + \beta p_1 - Tp_2^2 + \gamma Tp_1 - p_2^2 + \delta p_1 - p_2^2 + \max\{\varepsilon p_1 - Tp_1^2, \zeta p_2 - Tp_2^2\} \leq 0$$

And hence $(\alpha + \beta + \zeta) p_1 - p_2^2 \leq 0$.

We have from $(\alpha + \beta + \zeta) > 0$ that $p_1 = p_2$.

Therefore a fixed point theorem of T is unique. This completes the proof.

IV. REFERENCES

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