



On Consistency Issue of the Definition of Mean Free Path on the Basis of Measure Theoretic Probability and Speed of Light Postulate of Relativity

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Abstract: There are many essential as well as very basic derivations in physics which requires rigorous mathematical review and explanation and at the same time new theoretical foundation not only for mere sake of completeness, but for both aesthetic and practical need. One such topic in physics is Mean free path of kinetic theory of gases which has been addressed in this paper. The aim of this paper is to strengthen the concept of Mean free path by restating it in terms of Probability space and to use some tools of measure theory to show the consistency of the new definition with the old one and also to answer the basic question about whether it is well defined. It will also be shown here that the truth of speed of light postulate of special theory of relativity brings about the consistency of the definition of mean free path after Probability space is taken into account.

Keywords: Mean free path, Measure theory, Measurable functions, Dominated convergence theorem, Special theory of relativity, Probability space, Kinetic theory of gases.

I. Introduction

When a typical molecule moves through the gas, the path of it changes in both direction and speed because of collision with the other molecules. One of the very useful parameter to model this phenomenon is the mean free path λ of the molecules. Average distance traversed between collisions for a gas molecule is called the mean free path. But as the motion is random, uncertainty should be existing as an essential constituent in any model and parameter that tries to explain such type of motion. This motivates to bring the concept of Probability theory into picture.

II. Mean free path

Consider the motion of molecules in an ideal gas. Let n moles of an ideal gas be confined in a box of volume V , the same as was considered in Resnick [1]. The walls are maintained at temperature T . The pressure is p . If we concentrate now on path of a typical molecule as it moves through the gas, it collides elastically with other molecules which results in change in speed and direction. Needless to say that this motion is random. As stated in Resnick [1], one useful parameter to describe this random motion is the mean free path λ of the molecules, which is the average distance traversed by a molecule between collisions. Let the molecules have diameter d (without loss of generality). If N be the total number of molecules in that volume V (so that the number of molecules per unit volume is N/V), then the expression for the mean free path is: (Refer to Resnick [1])

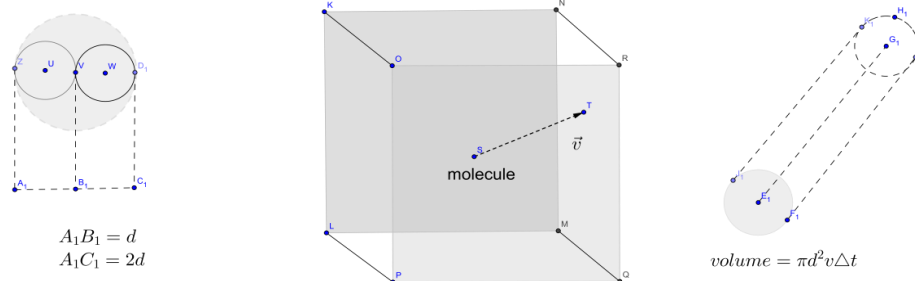
$$\lambda = \frac{1}{\sqrt{2}\pi d^2 \frac{N}{V}} \quad (1)$$

III. Necessity of Probabilistic model

As stated in the introduction part, the uncertainty of the motion of Gas molecule should itself require some well built uncertainty theory. Probability space is one of the current theory devoted to take care of uncertain phenomenon. The theory of Mean free path of kinetic theory of gases is still not rich in necessary but inevitable probability models. The analysis that exists in literature on this topic is somewhat imperfect and not beyond question of rigour. According to Feynman et al [3], one of the main reasons for which one is compelled to do the

analysis so called "imperfectly" is that the mathematics that it should require is a deep understanding of the theory of probability. Actually we are interested in the knowledge about how every atom is actually behaving, but we are compelled to view the whole thing as a gross, i.e., on the average. So perhaps this subject requires a demanding knowledge of probability and uncertainty theory, and it should be so demanding that we should have good reason to fear.

Figure 1: The assumed model



IV. Assumed Probabilistic Model

In this paper the simplest possible probabilistic model will be assumed with each molecules as follows. The notion of randomness will be incorporated in the theory by probability space, which is one kind of finite measure space. For simplicity, we will assume a probability space $(\Omega_t, \mathcal{A}_t, \mu_t)$ associated with each molecules which depends on time $t \in (0, \infty)$. Here it will be further assumed that the set of all such probability space is dominated by an universal probability measure space $(\Omega, \mathcal{A}, \mu)$ where μ dominates all of $\mu_t \forall t \in (0, \infty)$. To see what type of possibilities are coming in picture, consider Fig.1, where the molecule at position S in the box have velocity $\vec{v}$ at direction $\vec{ST}$. But the direction could have been any other and each such direction will give a Probable element in probability space.

V. Some Basic Results from Measure Theory

Measure theory is an aesthetic branch of mathematics which tries to generalize the sums and integrals that often comes overwhelmingly not only in calculus (one of the basic pillars of physics), but also in the distributions of discrete and continuous random variables and vectors. The generalization is through the concept of a measure, which is a way of assigning numerical values to the size of sets. (See Schervish [7]). One of the several basic results of measure theory is the following theorem. (For more details, see : Ash [6])

Theorem 5.1

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $a < b$ be any two real numbers.

(a) Suppose $f : (a, b) \times \Omega \mapsto \mathbb{R}$ such that : for each $t \in (a, b)$, the function $\omega \mapsto f(t, \omega)$ is $\mathcal{A}$ measurable and for each $\omega \in \Omega$, the function $t \mapsto f(t, \omega)$ is continuous on (a, b) . Let there exists a μ integrable measurable function g on Ω such that $|f(t, \omega)| \leq g(\omega) \forall (t, \omega)$, then $\varphi(t) = \int f(t, \omega) d\mu(\omega)$ defines a real-valued continuous function on (a, b) .

(b) Suppose $f : (a, b) \times \Omega \mapsto \mathbb{R}$ such that : $\forall t \in (a, b)$, $\omega \mapsto f(t, \omega)$ is $\mathcal{A}$ -measurable and μ - integrable, and $\forall \omega \in \Omega$, $t \mapsto f(t, \omega)$ is everywhere differentiable. Let $h(t, \omega) = \frac{\partial f(t, \omega)}{\partial t}$ and there be

a μ -integrable function g on Ω such that $|h(t, \omega)| \leq g(\omega) \forall (t, \omega)$. Then the function $\varphi(t) = \int f(t, \omega) d\mu(\omega)$ is everywhere differentiable and $\frac{d}{dt} \varphi(t) = \int h(t, \omega) d\mu(\omega)$.

Proof.

(a) Fix some $t \in (a, b)$. Also let t_n be any sequence such that $t_n \rightarrow t$. Then we have $f(t_n, \omega) \rightarrow f(t, \omega)$. Also we know, $|f(t, \omega)| \leq g(\omega)$. So, by Dominated Convergence Theorem we have $\varphi(t_n) \rightarrow \varphi(t)$, i.e. φ is continuous. Hence the proof.

(b) Fix some $t \in (a, b)$ and let $t_n \rightarrow t$. It is observed that $\frac{f(t_n, \omega) - f(t, \omega)}{t_n - t} = h(s_n, \omega)$, say, where s_n lies between t_n and t . So $h(s_n, \omega) \rightarrow h(t, \omega)$ (because $t_n \rightarrow t$). Now, as we know that $|h(s_n, \omega)| \leq g(\omega)$, we can apply Dominated Convergence Theorem to get that

$$\frac{\varphi(t_n) - \varphi(t)}{t_n - t} = \int \frac{f(t_n, \omega) - f(t, \omega)}{t_n - t} d\mu(\omega) = \int h(s_n, \omega) d\mu(\omega) \rightarrow \int h(t, \omega) d\mu(\omega).$$

Therefore, the function $\varphi(t) = \int f(t, \omega) d\mu(\omega)$ is everywhere differentiable and $\frac{d}{dt} \varphi(t) = \int h(t, \omega) d\mu(\omega)$. Hence the proof.

VI. The speed of light postulate of special relativity

One can restate the second postulate of special theory of relativity as: There is in nature an ultimate speed C , the same in all directions and in all inertial frames. Light happens to travel at this ultimate speed. No entity that carries energy or information can exceed this limit. Moreover, no particle that does have mass can actually reach speed C , regardless of how much or how long it is accelerated. The postulates have been exhaustively tested, and there have been no exceptions ever found (Reference: Resnick [1]). As it will be evident in the coming derivation, this second postulate will provide a sufficient condition for maintaining the consistency of the new definition of mean free path after considering the corresponding Probability space and measure.

VII. The problem in the derivation of mean free path

The derivation of mean free path can be found in Resnick [1], and one can find also in Serway [5] and [4], which is indeed a good yet very simple and obvious visualization. An equivalent brief overview of the derivation will be given below for the sake of completeness. The point is, the derivation is not mathematically rigorous and not beyond question of consistency if probability space is taken into picture (which is absolutely necessary, as stated previously). The main observation towards the derivation is the fact that a collision will take place if the centers of the molecules come within a distance d of each other. No two molecules can collide unless their centers are less than a distance d apart as they come towards each other. A single molecule covers out a short cylinder of cross sectional area of πd^2 between successive collisions. A way to model collisions is to imagine that one of the molecules has a diameter $2d$ and that the rest others are mere geometrical points (See Fig. 1). In time interval Δt , the molecule sweeps out a cylinder of cross-sectional area πd^2 and of length $v\Delta t$, where v is the assumed speed. (See Fig. 1). The volume of the cylinder is $\pi d^2 v\Delta t$. The number of collision that occur in time Δt is the number of molecules that lie within this cylinder and is equal to $(N/V)(\pi d^2 v\Delta t)$.

So, the mean free path $\lambda = \frac{v\Delta t}{\pi d^2 v\Delta t N/V} = \frac{1}{\pi d^2 N/V}$, as seen in equation(1), except the fact that it differs by a factor of $1/\sqrt{2}$ coming from the need of taking care of the assumption that all the molecules except one are rest (Refer to Resnick [1]). $\frac{1}{\sqrt{2}}$ is coming from the need of changing the velocity to the average relative velocity, and the calculation of that velocity from the molecular speed distribution yields average relative velocity $= \sqrt{2}v$. Hence the number of collisions is $\sqrt{2}$ times that of occurrence due to stationary molecules.

Now let us review the whole model again, but this time with the probability space, which is essential even because of the inner probabilistic structure of the assumed model, as stated by Feynman et al [3]. As it will be seen, some problem will be arising mainly due to consistency issues of definition, but it will be dissolved. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. (as defined before) Let $f \equiv f(t, \omega)$ denote length of the path upto time t . Here ω denote the one of the several Probable length of the path that can happen in probabilistic consideration.

Let $\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) = \int v(t, \omega) d\mu(\omega) \equiv \int v d\mu$ denote the expected value of v (the magnitude of $\vec{v}$),

where the expectation is being taken over all Probable velocities over Ω . As $\lambda = \frac{1}{\sqrt{2}\pi d^2 \frac{N}{V}}$, λ should be

same for all Probability space if N, V are same. One can restate the definition of mean free path with the fact that length of path is should be the average length of path, (where average is taken over elements of probability space) as follows: Mean free path, $\Lambda = \{ \text{length of mean (w.r.t } \mu) \text{ path during } \Delta t \} / \{ \text{number of mean(w.r.t } \mu) \text{ collisions in } \Delta t \}$

$$= \frac{\left| \int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega) \right|}{\frac{N}{V} \pi d^2 \left[\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right] \Delta t}.$$

Now the question is, whether $\Lambda = \lambda$ or not? It should be so to justify the question of whether the term is well defined or not. But we will see below that without the second assumption of Special Relativity theory (which had nothing to with our model and derivation so far obtained), in general not true. In other words, without the assumption of second axiom of special theory of relativity, the definition of mean path has its inner inconsistency with probability space. Relativity should come to picture to maintain consistency.

It is observed that :

$$\frac{\left| \int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega) \right|}{\Delta t} = \int \left(\frac{f(t + \Delta t, \omega) - f(t, \omega)}{\Delta t} \right) d\mu(\omega) \neq \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right)$$

That is, the last inequality will hold in general. As an immediate, we can impose condition that f should be differentiable, and then we will take Δt to zero. Then will it be true that

$$\lim_{\Delta t \rightarrow 0} \int \left(\frac{f(t + \Delta t, \omega) - f(t, \omega)}{\Delta t} \right) d\mu(\omega) = \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right) \quad ?$$

At this very point we have to take the help of relativity to show that it is true under some assumptions.

Observation 7.1 Let c be the velocity of light. From the assumption of Special Theory of Relativity we know

$$\text{that :} \quad \frac{\partial}{\partial t} f(t, \omega) \leq c(\omega) \quad (2)$$

It should not matter even if we assume that magnitude of velocity of light is different in different probable worlds. The first assumption of Relativity will not break at all, as it say nothing about the velocity.

Then, by the theorem stated in the section of measure theory, we can say that :

$$\lim_{\Delta t \rightarrow 0} \int \left(\frac{f(t + \Delta t, \omega) - f(t, \omega)}{\Delta t} \right) d\mu(\omega) = \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right),$$

But there is a bigger problem. This answer to the question raised just now is still not very satisfactory due to the following observation:

Observation 7.2 If $\Delta t \rightarrow 0$, then no. of collision goes to zero. In fact as number of collision is a non negative integer, so if for $\varepsilon > 0$ is such that $|\Delta t| < \varepsilon \Rightarrow \Delta N < \delta$, and ΔN is decreasing function of Δt , so there will be an $\varepsilon^*, \varepsilon$ such that $\Delta N = 0 \forall t \in (0, \varepsilon^*)$. The definition of mean free path will not hold good in those cases in the sense that it will not hold the physical motivation behind the birth of such a definition! Note that previously these type of issues had not come into picture as in previous derivation, $v\Delta t$ would cancel in the defining fraction for λ for any suitable Δt , we did not need to take limit.

The aim of this paper is to show that, in this case also second assumption of relativity is sufficient to maintain this consistency issue.

From the previous definition of mean free path :
$$\Lambda = \frac{\left[\int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega) \right]}{\frac{N}{V} \pi d^2 \left[\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right] \Delta t}$$

One can restate the definition as follows :
$$\Lambda = \frac{\left[\int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega) \right]}{\frac{N}{V} \pi d^2 \left[\int_t^{t+\Delta t} \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right) dt \right]}$$

Note that in the denominator the term $\left[\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right] \Delta t$ is now replaced by $\left[\int_t^{t+\Delta t} \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right) dt \right]$ in the new definition of Λ , which makes complete sense and has proper physical interpretation also. Now from equation (2) and the results of measure theory we can write:

$$\int_t^{t+\Delta t} \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right) dt = \int_t^{t+\Delta t} \frac{d}{dt} \left(\int f(t, \omega) d\mu(\omega) \right) dt = \int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega).$$

Hence after taking care of the assumption that all the molecules except one are rest, we have got $\Lambda = \lambda$, i.e.,

$$\Lambda = \frac{\left[\int f(t + \Delta t, \omega) d\mu(\omega) - \int f(t, \omega) d\mu(\omega) \right]}{\frac{N}{V} \pi d^2 \left[\int_t^{t+\Delta t} \left(\int \frac{\partial}{\partial t} f(t, \omega) d\mu(\omega) \right) dt \right]} = \frac{1}{\pi d^2 \frac{N}{V}}.$$

After making the correction of relative velocity (same as was done in previous derivation, see Resnick [1]),

$$\Lambda = \frac{1}{\sqrt{2} \pi d^2 \frac{N}{V}}, \text{ as desired. Hence now it is well defined.}$$

VI. Conclusion

Whether it is mathematics or physics, the rigour and perfection of mathematics should be taken care of at each steps. The probabilistic model that has been considered here is very simple in nature, and the simplicity should not be an adequate one to be able to take care of the whole complex random phenomenon that is going on. It is also shown here that the second postulate of relativity is required as a sufficient condition for maintaining the mathematical consistency. In other words, in reality the fact that there is a supreme velocity which we cannot cross is also mathematically a sufficient condition to maintain consistency in many situations.

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