



Theorems and General Inverse Value Approximation Formulae of Basic Trigonometric Functions

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Abstract: This paper presents study continuing from the previous published paper of title some applicable methods of approximating basic trigonometric function and their inverse value. The results in the previous publication help to derive the results included in this paper. The results of this paper are theorem of approximating value, general inverse value approximation formulae and theorem of locating inverse value within the known interval. The objective is to contribute additional knowledge to the areas of Mathematical science.

Key word: sine function, cosine function, co function definition, sum formula

I. Introduction

The previous publication discusses about the some applicable methods of approximating basic trigonometric functions and their inverse value. Some of those methods are mentioned below [1].

A. Method One

If the A, B, ... stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.AB * 60 + 360n) = \pm 0.AB \dots \text{if } 0 \leq 0.AB \leq 0.50$$

$$\cos(\pm(90 - 0.AB * 60) + 360n) = 0.AB \dots \text{if } 0 \leq 0.AB \leq 0.50$$

Where n is integer

The maximum error of approximation is less than 0.009

B. First Approximation Method

$$\sin(x + 360n) \approx \frac{1}{60}x \text{ if } -30^\circ \leq x \leq 30^\circ$$

$$\cos(90 - x + 360n) \approx \frac{1}{60}x \text{ if } -30^\circ \leq x \leq 30^\circ$$

The maximum error of approximation is less than 0.009

C. Method Two

If the A, B, C stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.ABC * \frac{60}{\sqrt{6}-\sqrt{2}} + 360n) \approx \begin{cases} \pm 0.ABC & \text{if } 0 \leq 0.ABC \leq 0.258 \text{ and } A \neq 1 \\ \pm(0.ABC + 0.001) & \text{if } 0 \leq 0.ABC \leq 0.258 \text{ and } A = 1 \end{cases}$$

$$\cos(\pm(90 - 0.ABC * \frac{60}{\sqrt{6}-\sqrt{2}}) + 360n) \approx \begin{cases} 0.ABC & \text{if } 0 \leq 0.ABC \leq 0.258 \text{ and } A \neq 1 \\ 0.ABC + 0.001 & \text{if } 0 \leq 0.ABC \leq 0.258 \text{ and } A = 1 \end{cases}$$

The maximum error of approximation is less than 0.0009

D. Second Approximation Method

$$\sin(x + 360n) \approx \frac{\sin 15}{15}x \approx \frac{\sqrt{6}-\sqrt{2}}{60}x \text{ if } -15^\circ \leq x \leq 15^\circ$$

$$\cos(90 - x + 360n) \approx \frac{\sin 15}{15}x \approx \frac{\sqrt{6} - \sqrt{2}}{60}x \text{ if } -15^\circ \leq x \leq 15^\circ$$

The maximum error of approximation is less than 0.0009

Sine and cosine functions related by the following trigonometric identity [2]

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

Co function definition for sine and cosine function given by [3]

$$\sin \theta = \cos(90 - \beta)$$

In other word, if $\theta + \beta = 90^\circ$, then $\sin \theta = \cos \beta$

Sum of two angle formula given by [2]

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

II. Materials and Methods

A. Theorem of Approximating Value

If the β is angle within the interval $-90^\circ \leq \beta \leq 90^\circ$, then the following approximate equations hold true.

$$\sin(\beta + 360n) \approx 2\sqrt{1 - x^2}(1 - 4x^2)(3x - 4x^3)$$

$$\cos(90 - \beta + 360n) \approx 2\sqrt{1 - x^2}(1 - 4x^2)(3x - 4x^3)$$

Where n is integer

$$x \approx \frac{\sqrt{6} - \sqrt{2}}{360} \beta$$

Proof

Consider $0 \leq \beta \leq 90^\circ$

$$\text{Let } \beta \approx 6\theta \quad (1)$$

$$\Rightarrow \theta = \frac{\beta}{6} \quad (2)$$

$$\Rightarrow 0 \leq \theta \leq 15^\circ$$

Consider the second approximation method

$$\sin \theta \approx \frac{\sqrt{6} - \sqrt{2}}{60} \theta \text{ if } 0 \leq \theta \leq 15^\circ \quad (3)$$

Consider method two

$$\sin(0. \text{ABC} * \frac{60}{\sqrt{6} - \sqrt{2}}) \approx 0. \text{ABC} \text{ if } 0 \leq 0. \text{ABC} \leq 0.258 \quad (4)$$

Expand $\sin 6\theta$ by applying the sum of two angle formula

$$\sin 6\theta = 2 \cos \theta (1 - 4\sin^2 \theta)(3 \sin \theta - 4\sin^3 \theta) \quad (5)$$

Consider the following Trigonometric identity

$$\cos \theta = \sqrt{1 - \sin^2 \theta} \quad (6)$$

Substitute equation (6) into (5)

$$\sin 6\theta = 2\sqrt{1 - \sin^2 \theta}(1 - 4\sin^2 \theta)(3 \sin \theta - 4\sin^3 \theta) \quad (7)$$

Correlate equation (1) and (7)

$$\sin 6\theta = \sin \beta = 2\sqrt{1 - \sin^2 \theta}(1 - 4\sin^2 \theta)(3 \sin \theta - 4\sin^3 \theta) \quad (8)$$

Consider equation (4)

$$\sin(0. \text{ABC} * \frac{60}{\sqrt{6} - \sqrt{2}}) \approx 0. \text{ABC} \text{ if } 0 \leq 0. \text{ABC} \leq 0.258 \quad (9)$$

$$\text{Let } \theta \approx 0. \text{ABC} * \frac{60}{\sqrt{6} - \sqrt{2}} \quad (10)$$

Substitute equation (10) into (9)

$$\sin \theta \approx 0. \text{ABC} \text{ if } 0 \leq 0. \text{ABC} \leq 0.258 \quad (11)$$

Rearrange equation (10)

$$0. \text{ABC} \approx \frac{\sqrt{6} - \sqrt{2}}{60} \theta \quad (12)$$

Substitute equation (2) into (12)

$$0. \text{ABC} \approx \frac{\sqrt{6} - \sqrt{2}}{360} \beta$$

Substitute equation (11) into (8)

$$\sin \beta \approx 2\sqrt{1 - (0. \text{ABC})^2}(1 - 4(0. \text{ABC})^2)(3 * 0. \text{ABC} - 4(0. \text{ABC})^3)$$

Let $x = 0. \text{ABC}$

$$\sin \beta \approx 2\sqrt{1 - x^2}(1 - 4x^2)(3x - 4x^3) \text{ if } x \approx \frac{\sqrt{6} - \sqrt{2}}{360} \beta$$

It is proved that this equation holds true for $0 \leq \beta \leq 90^\circ$ and make sure that this equation also holds true for $-90 \leq \beta \leq 90^\circ$.

Apply co function definition. Therefore, $\cos(90 - \beta + 360n) \approx 2\sqrt{1-x^2}(1-4x^2)(3x-4x^3)$

B. General Inverse Value Approximation Formulae

Steps to derive angle approximation formulae

Consider the first approximation method/for positive angles/

$$\sin \theta \approx \frac{\theta}{60} \text{ if } 0 \leq \theta \leq 30^\circ \quad (13)$$

Consider three cases

$$\text{a) Let } \theta = \beta \quad (14)$$

Substitute equation (14) into (13)

$$\sin \beta \approx \frac{\beta}{60} \text{ if } 0 \leq \beta \leq 30^\circ \quad (15)$$

$$\text{b) Let } \theta + \beta = 90^\circ \text{ and } 0 \leq \theta \leq 30^\circ$$

$$\Rightarrow \theta = 90 - \beta \quad (16)$$

$$\Rightarrow \beta = 90 - \theta$$

$$\Rightarrow 60^\circ \leq \beta \leq 90^\circ$$

Consider Trigonometric identity

$$\sin^2 \beta = 1 - \cos^2 \beta \quad (17)$$

If $\theta + \beta = 90^\circ$, then the equation (18) holds true

$$\Rightarrow \cos \beta = \sin \theta \quad (18)$$

Substitute equation (18) into (17)

$$\sin^2 \beta = 1 - \sin^2 \theta \quad (19)$$

Substitute equation (13) into (19)

$$\begin{aligned} \sin^2 \beta &\approx 1 - \left(\frac{\theta}{60}\right)^2 \\ \Rightarrow \sin \beta &\approx \frac{\sqrt{3600-\theta^2}}{60} \end{aligned} \quad (20)$$

Substitute equation (16) into (20)

$$\sin \beta \approx \frac{\sqrt{3600-(90-\beta)^2}}{60} \text{ if } 60^\circ \leq \beta \leq 90^\circ \quad (21)$$

$$\text{c) Let } \beta = 30 + \theta \text{ and } 0 \leq \theta \leq 30^\circ$$

$$\Rightarrow \theta = \beta - 30 \quad (22)$$

$$\Rightarrow 30 \leq \beta \leq 60$$

Expand $\sin(30 + \theta)$ by applying the sum of two angle formula

$$\sin \beta = \sin(30 + \theta) = \sin 30 \cos \theta + \sin \theta \cos 30$$

$$\sin \beta = \sin(30 + \theta) = \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \quad (23)$$

Consider trigonometric identity

$$\cos \theta = \sqrt{1 - \sin^2 \theta} \quad (24)$$

Substitute equation (24) into (23)

$$\sin \beta = \frac{1}{2} \sqrt{1 - \sin^2 \theta} + \frac{\sqrt{3}}{2} \sin \theta \quad (25)$$

Substitute equation (13) into (25)

$$\begin{aligned} \sin \beta &\approx \frac{1}{2} \sqrt{1 - \left(\frac{\theta}{60}\right)^2} + \frac{\sqrt{3}}{2} * \frac{\theta}{60} \\ \sin \beta &\approx \frac{\sqrt{3600-\theta^2} + \sqrt{3}\theta}{120} \end{aligned} \quad (26)$$

Substitute equation (22) into (26)

$$\sin \beta \approx \frac{\sqrt{3600-(\beta-30)^2} + \sqrt{3}(\beta-30)}{120} \text{ if } 30^\circ \leq \beta \leq 60^\circ \quad (27)$$

Assembly equations (15), (21) and (27) altogether;

$$\sin \beta \approx \begin{cases} \frac{\beta}{60} & \text{if } 0 \leq \beta \leq 30^\circ \\ \frac{\sqrt{3600-(\beta-30)^2} + \sqrt{3}(\beta-30)}{120} & \text{if } 30^\circ \leq \beta \leq 60^\circ \\ \frac{\sqrt{3600-(90-\beta)^2}}{60} & \text{if } 60^\circ \leq \beta \leq 90^\circ \end{cases} \quad (28)$$

Express equation (28) in terms of β

$$\beta \approx \begin{cases} 60 \sin \beta & \text{if } 0 \leq \sin \beta \leq 0.50 \\ \frac{240(1+\sqrt{3} \sin \beta) - \sqrt{240^2(1+\sqrt{3} \sin \beta)^2 - 115200 \sin \beta (\sqrt{3} + 2 \sin \beta)}}{8} & \text{if } \frac{1}{2} \leq \sin \beta \leq \frac{\sqrt{3}}{2} \\ 90 - 60\sqrt{1 - \sin^2 \beta} & \text{if } \frac{\sqrt{3}}{2} \leq \sin \beta \leq 1 \end{cases}$$

If reference angle, co-terminal angles and negative value of $\sin \beta$ are taken in to consideration, then the inverse value of sine function for any given value can be calculated by

$$\beta \approx \begin{cases} 360x - m(90 \pm 60 |\sin \beta|) + 360n & \text{if } 0 \leq |\sin \beta| \leq 0.50 \\ 360x - m(90 \pm \frac{240(1+\sqrt{3} |\sin \beta|) - \sqrt{240^2(1+\sqrt{3} |\sin \beta|)^2 - 115200 |\sin \beta| (\sqrt{3} + 2 |\sin \beta|)}}{8}) + 360n & \text{if } \frac{1}{2} \leq |\sin \beta| \leq \frac{\sqrt{3}}{2} \\ 360x - m(90 \pm 60\sqrt{1 - \sin^2 \beta}) + 360n & \text{if } \frac{\sqrt{3}}{2} \leq |\sin \beta| \leq 1 \end{cases}$$

Where n is integer

x = 0 and m = -1 if $\sin \beta$ is positive

x = 1 and m = 1 if $\sin \beta$ is negative

Apply trigonometric identity and therefore, Inverse value of cosine function can be calculated by

$$\beta \approx \begin{cases} 180x + 360n \pm 60\sqrt{1 - \cos^2 \beta} & \text{if } \frac{\sqrt{3}}{2} \leq |\cos \beta| \leq 1 \\ 180x + 360n \pm \frac{240(1-|\cos \beta|) + \sqrt{240^2(1-|\cos \beta|)^2 - 115200(2\cos^2 \beta - |\cos \beta| - 1)}}{8} & \text{if } \frac{1}{2} \leq |\cos \beta| \leq \frac{\sqrt{3}}{2} \\ 180x + 360n \pm (90 - 60 |\cos \beta|) & \text{if } 0 \leq |\cos \beta| \leq \frac{1}{2} \end{cases}$$

Where n is integer

x = 0 if $\cos \beta$ is positive

x = 1 if $\cos \beta$ is negative

C. Theorem of Locating Inverse Value Within the Known Interval

This section locates inverse value within the interval having upper and lower bound difference 1.2°

If $\sin \theta = \pm 0.AB \dots$ and $0 \leq 0.AB \leq 0.50$, then

$$\pm 0.AB * 60 - 0.6 + 360n < \theta < \pm 0.AB * 60 + 0.6 + 360n$$

Where A and B stands for digits after a decimal point

Proof

Consider method two

$$\sin(\pm 0.AB * 60 + 360n) = \pm 0.AB \dots \text{if } 0 \leq 0.AB \leq 0.50 \quad (29)$$

$$\text{Let } \theta = \pm 0.AB * 60 + 360n \quad (30)$$

$$x = (\pm 0.AB - 0.01) * 60 + 360n \quad (31)$$

$$y = (\pm 0. AB + 0.01) * 60 + 360n \quad (32)$$

Correlate equations (30), (31) and (32)

$$x < \theta < y \quad (33)$$

$$y - x = 1.2^\circ$$

Substitute equations (31) and (32) into (33)

$$\begin{aligned} (\pm 0. AB - 0.01) * 60 + 360n < \theta < (\pm 0. AB + 0.01) * 60 + 360n \\ (\pm 0. AB * 60 - 0.6 + 360n) < \theta < (\pm 0. AB * 60 + 0.6 + 360n) \end{aligned} \quad (34)$$

Correlate equations (29) and (34)

If $\sin x = \pm 0. AB \dots$ and $0 \leq 0. AB \leq 0.50$, then

$$\pm 0. AB * 60 - 0.6 + 360n < \theta < \pm 0. AB * 60 + 0.6 + 360n$$

III. Results and Discussion

The results of this paper are theorem of approximating value, general inverse value approximation formulae and theorem of locating inverse value within the known interval. General inverse approximation formulae help to approximate angle for any given value.

Example1. Approximate θ if $\sin \theta = 0.967$

$$\theta \approx 360x - m(90 \pm 60\sqrt{1 - \sin^2\theta}) + 360n \text{ if } \frac{\sqrt{3}}{2} \leq |\sin \theta| \leq 1$$

Since $\sin \theta = 0.967$ is positive; therefore, $x = 0$ and $m = -1$

$$\theta \approx 90 \pm 60\sqrt{1 - \sin^2\theta} + 360n \text{ if } \frac{\sqrt{3}}{2} \leq |0.967| \leq 1$$

$$\theta \approx 90 \pm 60\sqrt{1 - 0.967^2} + 360n \text{ if } \frac{\sqrt{3}}{2} \leq 0.967 \leq 1$$

$$\theta \approx 105.29 + 360n \text{ Or } \theta \approx 74.71 + 360n$$

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