



## Generalizing the Concept of Membership Function of Fuzzy Sets on the Real line Using Distribution Theory

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**Abstract:** Membership function of a fuzzy set is the generalization of the characteristic function of a crisp set. The classical membership function is a function from a set to the interval  $[0,1]$ . In this paper we generalize the concept of membership function of fuzzy sets on the real line using 'generalized functions' or distributions.

**Keywords:** crisp set; characteristic function; fuzzy set; membership function; generalized function; distribution theory

### I. Introduction

The characteristic function of a crisp set is a function from a set to  $\{0, 1\}$ . In 1940 H. Weyl had thought of generalizing the concept of characteristic function by using any continuous function [4]. In 1965 Lofti A Zadeh introduced the concept of a fuzzy set [5]. The classical membership function of a fuzzy set is a function from a set to the interval  $[0, 1]$ . This has been generalized in several ways. For example an L-fuzzy set has a membership function where the range is a partially ordered set [2].

In this paper we generalize the concept of membership function of fuzzy sets on the real line using distribution theory. The membership function is not restricted to be an ordinary function. Instead it can be a 'generalized function' or a distribution.

It will be shown that this generalization has a definite advantage.

### II. Preliminaries

Distributions can be thought of as generalization of the usual functions ([1], [3]). We briefly give the relevant definitions here.

Notation:  $\mathbb{R}$  is the set of all real numbers.  $C_0^\infty(\mathbb{R})$  is the set of all functions from  $\mathbb{R}$  to  $\mathbb{R}$  which are infinitely differentiable and have compact support.

Definition 2.1:  $f(x)$  is said to be a locally integrable function if  $\int_a^b |f(x)|dx$  is defined and is finite for any finite interval  $[a,b]$ .

Definition 2.2: A functional is a mapping from  $C_0^\infty(\mathbb{R})$  to  $\mathbb{R}$ .

Definition 2.3: A functional  $T$  is said to be linear if  $T(mf+ng)=mT(f)+nT(g)$  for all  $m, n \in \mathbb{R}$  and  $f, g \in C_0^\infty(\mathbb{R})$ .

Definition 2.4: A sequence of smooth functions  $\{f_n\}$  in  $C_0^\infty(\mathbb{R})$  is said to converge to zero in  $C_0^\infty(\mathbb{R})$  if

- (i) for every  $k \in \mathbb{N} \cup \{0\}$ , the sequence of  $k^{\text{th}}$  derivatives  $f_1^k(x), f_2^k(x), \dots$  converges uniformly to zero.
- (ii) there is a common interval  $[a, b]$  independent of  $n$  such that every  $f_n(x)$  vanishes outside  $[a, b]$

Note: We say that a sequence  $\{f_n\}$  converges to  $f$  in  $C_0^\infty(\mathbb{R})$  if and only if the sequence  $\{f_n - f\}$  converges to zero in  $C_0^\infty(\mathbb{R})$ .

Definition 2.5: A functional  $T$  is said to be continuous if for any sequence  $\{g_n\}$  of smooth functions converging to  $g$  in  $C_0^\infty(\mathbb{R})$ ,  $\lim_{n \rightarrow \infty} T(g_n) = T(g)$

Note: A linear functional  $T$  is said to be continuous if for any sequence  $\{g_n\}$  in  $C_0^\infty(\mathbb{R})$  converging to zero in  $C_0^\infty(\mathbb{R})$ ,  $\lim_{n \rightarrow \infty} T(g_n) = 0$

Definition 2.6: A functional which is both linear and continuous is called a generalized function or a distribution.

Example 2.1: The functional  $\delta$  defined by  $\delta(g(x)) = g(0)$  is a distribution:

Clearly  $\delta$  is linear. For proving continuity let  $\{g_n\}$  be a sequence of functions in  $C_0^\infty(R)$  converging to zero in  $C_0^\infty(R)$ . Then  $\lim_{n \rightarrow \infty} \delta(g_n) = \lim_{n \rightarrow \infty} g_n(0) = 0$ .

$\delta$  is called delta distribution.

Example 2.2: Let  $f(x)$  be any locally integrable function. Define an operator  $T_f$  on  $C_0^\infty(R)$  by

$$T_f(g) = \int_{-\infty}^{+\infty} f(x)g(x)dx.$$

Clearly  $T_f$  is well defined and is a linear operator on  $C_0^\infty(R)$ .

Let  $\{g_n\}$  be a sequence of functions in  $C_0^\infty(R)$  converging to zero in  $C_0^\infty(R)$ . By the definition there exists an interval  $[a, b]$  such that  $g_n$  vanishes outside of  $[a, b]$  for all  $n$ .

Also the sequence  $\{g_n(x)\}$  converges uniformly to zero. So for any  $\varepsilon > 0$ ,  $\exists n_0 \in N$  such that

$$|g_n(x) - 0| < \frac{\varepsilon}{M} \quad \forall x \in [a, b], \forall n \geq n_0 \text{ where } M = \int_a^b |f(x)| dx.$$

$$\begin{aligned} |T_f(g_n)| &= \left| \int_a^b g_n(x)f(x)dx \right| \\ &\leq \int_a^b |g_n(x)| |f(x)| dx \\ &< \frac{\varepsilon}{M} \int_a^b |f(x)| dx \quad \forall n \geq n_0 \\ &= \varepsilon / M \cdot M \\ &= \varepsilon \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} T_f(g_n) = 0$$

Hence  $T_f$  is a distribution.

Definition 2.7: If  $f$  is a locally integrable function then the distribution  $T_f$  defined by

$$T_f(g) = \int_{-\infty}^{+\infty} f(x)g(x)dx$$

is said to be induced by the function  $f$  and such distributions are said to be regular.

Definition 2.8: A distribution which is not induced by any locally integrable function is said to be singular.

Example 2.3: It is well known that the delta distribution is a singular distribution.

Definition 2.9: A function  $g$  from  $R$  to  $R$  is said to be non negative if  $g(x) \geq 0 \quad \forall x \in R$ . We write  $g \geq 0$  in this case.

Definition 2.10: A distribution  $T$  is said to be positive if  $T(g) \geq 0$  whenever  $g \geq 0$ .

### III. Distributed Fuzzy Set

Consider a sequence of functions  $\{f_n\}$  in  $C_0^\infty(R)$  with the following properties with respect to the interval  $[a, b]$ .

$$P1 : f_n(x) = 0 \quad \text{out side } [a - 1/n, b + 1/n].$$

$$P2 : f_n(x) = 1 \quad \text{in } [a + 1/n, b - 1/n].$$

$$P3 : 0 \leq f_n(x) \leq 1.$$

It can be shown that such a sequence exists.

Example 3.1:

$$\text{For } x \in R, \text{ define } h(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

Then  $h$  is infinitely differentiable and its  $n^{\text{th}}$  derivative has the form

$$h^{(n)}(x) = P_n\left(x, \frac{1}{1-x^2}\right)h(x) \quad \forall n.$$

where  $P_n(x, y)$  is a polynomial in  $x, y$ .

Also support of  $h$  is  $[-1, 1]$  which is compact.

$$\therefore h \in C_0^\infty(R)$$

Let  $g(x) = \int_{-\infty}^x h(t)dt$ . Then  $g$  is infinitely differentiable and

$$g(x) = \begin{cases} 0 & \text{for } x < -1 \\ k & \text{for } x \geq 1 \end{cases}$$

$$\text{And } 0 \leq g(x) \leq k \quad \forall x$$

For any  $\lambda > 0$ , define  $g_\lambda(x) = g(\lambda x)$

$$\text{Let } g_{\lambda,a}(x) = g(\lambda(x-a))$$

$$g_{\lambda,a}(x) = \begin{cases} 0 & x < a - 1/\lambda \\ k & x \geq a + 1/\lambda \end{cases}$$

$$\text{And } 0 \leq g_{\lambda,a}(x) \leq k$$

Let  $a < b$  and let  $\lambda$  and  $\mu$  be such that  $a < a + 1/\lambda < b - 1/\mu < b$ .

Let  $f(x) = g_{\lambda,a}(x) g_{\mu,-b}(-x) = g(\lambda(x-a))g(\mu(b-x))$

Then  $f$  is infinitely differentiable.

$$\text{Also } f(x) = \begin{cases} 0 & \text{if } x \leq a - 1/\lambda \text{ or } x \geq b + 1/\mu \\ k^2 & \text{if } a + 1/\lambda \leq x \leq b - 1/\mu \end{cases}$$

$$0 \leq f(x) \leq k^2 \text{ and support of } f \subseteq [a - 1/\lambda, b + 1/\mu].$$

$$\therefore f \in C_0^\infty(R)$$

$$\text{Now consider } f_n(x) = \frac{1}{k^2} g_{n,a}(x)g_{n,-b}(-x) \text{ for } n > \frac{2}{b-a}$$

Then each  $f_n \in C_0^\infty(R)$  and we have

$$f_n(x) = \begin{cases} 0 & \text{outside } [a - 1/n, b + 1/n] \\ 1 & \text{in } [a + 1/n, b - 1/n] \end{cases}$$

$$\text{And } 0 \leq f_n(x) \leq 1$$

**Definition 3.1:** Let  $\{f_n\}$  be a sequence of functions in  $C_0^\infty(R)$  satisfying P1, P2, and P3 with respect to the interval  $[a, b]$  and let  $T$  be a distribution.  $T$  is said to define a distributed fuzzy set on  $[a, b]$  if

- i)  $T$  is a positive distribution
- ii)  $\lim_{n \rightarrow \infty} T(f_n)$  exists and is the same for any sequence of functions  $\{f_n\}$  in  $C_0^\infty(R)$  satisfying the P1, P2, and P3 with respect to the interval.
- iii)  $\lim_{n \rightarrow \infty} T(f_n) \leq b-a$ .

**Example 3.2:** In example 2.1 it was shown that  $\delta$  is a distribution. Also if  $g$  is non negative then

$$\delta(g(x)) = g(0) \geq 0$$

$\delta$  is a positive distribution.

Consider a sequence of functions  $\{f_n\}$  in  $C_0^\infty(R)$  with the properties P1, P2 and P3 with respect to  $[-1, 1]$ .

Note that for  $n \geq 2$ ,  $0 \in [-1 + 1/n, 1 - 1/n]$

$$\delta(f_n) = f_n(0) = 1 \quad \forall \quad n \geq 2$$

$$\lim_{n \rightarrow \infty} \delta(f_n) = 1 < 1 - (-1)$$

$\therefore \delta$  defines a distributed fuzzy set on  $[-1, 1]$ .

**Example 3.3:** Consider the distribution  $T_\mu$ :

$$T_\mu(f(x)) = \int_{-\infty}^{+\infty} \mu(x)f(x) dx$$

where  $\mu(x)$  is locally integrable and is the member ship function of a fuzzy set. That is  $\mu(x)$  is such that (i)

$$\int_c^d \mu(x) dx \text{ is finite for any interval } [c, d] \quad \text{(ii) } 0 \leq \mu(x) \leq 1 \quad \forall x \in R.$$

Clearly  $T_\mu$  is a positive distribution.

Let  $\{f_n\}$  be a sequence of functions in  $C_0^\infty(R)$  satisfying the P1, P2, and P3.

$$\begin{aligned} \left| T_\mu(f_n) - \int_a^b \mu(x) dx \right| &= \left| \int_{-\infty}^{+\infty} \mu(x)f_n(x)dx - \int_a^b \mu(x) dx \right| \\ &= \left| \int_{a-1/n}^{b+1/n} \mu(x)f_n(x)dx - \int_a^b \mu(x) dx \right| \\ &= \left| \int_{a-1/n}^a \mu(x)f_n(x)dx + \int_a^b \mu(x)(f_n(x) - 1)dx + \int_b^{b+1/n} \mu(x)f_n(x)dx \right| \\ &\leq \left| \int_{a-1/n}^a \mu(x)f_n(x)dx \right| + \left| \int_a^b \mu(x)(f_n(x) - 1)dx \right| + \left| \int_b^{b+1/n} \mu(x)f_n(x)dx \right| \end{aligned} \quad (1)$$

$$\begin{aligned} \left| \int_{a-1/n}^a \mu(x)f_n(x)dx \right| &\leq \int_{a-1/n}^a |\mu(x)||f_n(x)| dx \\ &\leq \left( a - a + \frac{1}{n} \right) \\ &= 1/n \end{aligned} \quad (2)$$

$$\left| \int_a^b \mu(x)(f_n(x) - 1)dx \right| \leq \int_a^b |\mu(x)||f_n(x) - 1| dx$$

$$\begin{aligned} &\leq \int_a^{a+1/n} |\mu(x)| |(f_n(x) - 1)| dx + \int_{b-1/n}^b |\mu(x)| |(f_n(x) - 1)| dx \\ &\leq 2(a + 1/n - a) + 2(b - b + 1/n) \\ &= 4/n \end{aligned} \quad (3)$$

$$\begin{aligned} \left| \int_b^{b+1/n} \mu(x) f_n(x) dx \right| &\leq \int_b^{b+1/n} |\mu(x)| |f_n(x)| dx \\ &\leq (b + 1/n - b) \\ &= 1/n \end{aligned} \quad (4)$$

From (1), (2), (3) and (4)

$$\left| T_\mu(f_n) - \int_a^b \mu(x) dx \right| \leq \frac{6}{n} < \epsilon \quad \forall n \geq \frac{6}{\epsilon}$$

$$\lim_{n \rightarrow \infty} T_\mu(f_n) = \int_a^b \mu(x) dx$$

( since  $0 \leq \mu(x) \leq 1$  )

Hence  $T_\mu$  defines a distributed fuzzy set on  $[a, b]$ .

Example 3.3 is the motivation behind the following definition:

Definition 3.2: Suppose  $T$  defines a distributed fuzzy set over a interval  $[a, b]$ . Then it is said to have average membership value  $m$  over the interval  $[a, b]$  where

$$m = \frac{1}{b-a} \lim_{n \rightarrow \infty} T(f_n)$$

Theorem :

Suppose  $T$  defines a distributed fuzzy set over the interval  $[a, x]$  for every  $x$  such that  $a \leq x \leq b$  and  $m(x)$  is the average membership value of the distributed fuzzy set over the interval  $[a, x]$ . Suppose  $m(x)$  is continuously differentiable in the interval  $[a, b]$ . If  $T_\mu$  is the regular distribution induced by  $\mu(x)$  where

$$\mu(x) = \begin{cases} \frac{d}{dx}((x-a)m(x)) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

then  $T_\mu$  defines a distributed fuzzy set over the interval  $[a, x]$  for every  $x$  such that  $a \leq x \leq b$  and the average membership value of this distributed fuzzy set over the interval  $[a, x]$  is also  $m(x)$ .

Proof: It is clear that  $\mu(x)$  is locally integrable. So  $T_\mu$  defines a distributed fuzzy set over the interval  $[a, x]$  for every  $x$  such that  $a \leq x \leq b$ . Let  $A(x)$  denote the average membership value of the distributed fuzzy set defined by  $T_\mu$  in the interval  $[a, x]$ . In example 3.3 it was shown

$$\lim_{n \rightarrow \infty} T_\mu(f_n) = \int_a^x \mu(t) dt$$

if  $\{f_n\}$  is a sequence of functions in  $C_0^\infty(R)$  satisfying the P1, P2, and P3 with respect to the interval  $[a, x]$ . Hence

$$\begin{aligned} A(x) &= \frac{1}{x-a} \lim_{n \rightarrow \infty} T_\mu(f_n) = \frac{1}{x-a} \int_a^x \mu(t) dt \\ &= \frac{1}{x-a} \int_a^x \frac{d}{dt}((t-a)m(t)) dt \\ &= \frac{1}{x-a} ((x-a)m(x)) \\ &= m(x) \end{aligned}$$

Note: In other words if the hypothesis of the theorem is satisfied then we can consider the distributed fuzzy set to be defined by a regular distribution  $T_\mu$ . Also in this case we have the usual fuzzy set defined on the interval  $[a, b]$  with the membership function  $\mu(x)$ .

#### IV. Conclusion

The example 3.3 shows that a fuzzy set on the real line is a distributed fuzzy set provided the membership function is locally integrable. It is known that [2] the membership function of a fuzzy number or fuzzy interval is of the form

$$\mu(x) = \begin{cases} l(x) & \text{for } x \in (-\infty, a) \\ 1 & \text{for } x \in [a, b] \\ r(x) & \text{for } x \in (b, \infty) \end{cases}$$

where  $l(x)$  is a function from  $(-\infty, a)$  to  $[0, 1]$  that is monotonic increasing, continuous from the right and such that  $l(x) = 0$  in  $(-\infty, u)$ ; where  $r(x)$  is a function from  $(b, \infty)$  that is monotonic decreasing, continuous from the left and such that  $r(x) = 0$  in  $(v, \infty)$ . It is clear that the above  $\mu(x)$  is locally integrable. So fuzzy numbers and fuzzy intervals are distributed fuzzy sets.

Conversely suppose we have a distributed fuzzy set on  $[a, b]$  defined by a distribution  $T$ . Suppose the average membership value  $m(x)$  over every subinterval  $[a, x]$  can be determined for every  $x$  in  $[a, b]$ . If  $m(x)$  is continuously differentiable then as we have already remarked this distributed fuzzy set corresponds to the usual fuzzy set. In this case the membership function  $\mu(x)$  is determinable at every point in the interval  $[a, b]$ .

However there are distributions where this is not possible. Example 3.2 is one such case. In other words only the average membership value over a certain interval can be determined and within this interval we cannot specify the value of the membership more precisely. This is inbuilt impreciseness or vagueness. Hence the concept of distributed fuzzy set inherently models vagueness.

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