



## EFFECT OF SUPPORT CONDITION ON NUMERICAL FACTORS FOR DEFLECTION OF A CENTRALLY LOADED THIN RECTANGULAR ISOTROPIC PLATE

Emma J.B.<sup>1</sup>, Sule, S.<sup>2</sup>

<sup>1</sup>Department of Civil Engineering  
University of Nigeria, Nsukka, Enugu State, Nigeria  
<sup>2</sup>Department of Civil and Environmental Engineering,  
University of Port Harcourt, Rivers State, Nigeria

**Abstract:** In this paper, the values numerical factors for deflection of a centrally loaded thin rectangular isotropic plate subjected to different support conditions: all round fixed, all round simply supported, two opposite sides fixed and the other two simply supported are determined using energy approach. For the three support conditions, it was found that the numerical factor for deflection increased with increase in aspect ratio of plate up to 1.3. It was also found that an all round simply supported plate has the highest values of numerical factors for deflection followed by two opposite sides fixed and the other sides simply supported and then all round fixed plate.

**Keywords:** numerical factors, rectangular isotropic plate, support conditions, all round fixed plate, energy approach.

### I. Introduction

A plate as an engineering structure is defined as a body in the shape of a prism with thickness, small compared with its other two dimensions. They are commonly referred to as thin-walled structures. Plates are used in modern structures to transmit lateral and or in-plane load to adjacent supports. Engineering plates are employed in engineering construction such as, building, civil engineering, hydraulic engineering, naval architecture and air-craft construction [1-5]. The classical method that leads to exact solution is not only rigorous and time consuming but proves in many cases quite laborious and mathematically cumbersome [6]. The problems encountered in thin plate theory can be solved with the aid of various approximate methods such as energy method, finite element method, finite difference method and fourier series [7]. However, common problems are encountered. For example, numerical approaches such as finite difference and finite element method lead to an algebraic equation of large matrix size which requires large computer memories, thereby making the analysis cumbersome and time wasting. Navier's method is regarded as the widely used approximate method for the analysis of thin plates. However, researches have shown that the double trigonometric series in the method are not convenient for numerical computations if higher derivatives of the function  $w$  are involved. Besides, satisfactory solution by Navier method is only obtained for simply supported thin plates. In this paper, the polynomial function is employed in the estimation of numerical factors for deflection of a thin rectangular isotropic plate subjected to a centre point loading considering three different boundary conditions.

From statics, the potential energy of applied load is given by:

$$V = -pw(x, y) \quad (1)$$

and the total potential energy of the system is given by:

$$\pi_T = U + V \quad (2)$$

From equations (1) and (2) the total potential energy of the system is given by:

$$\pi_T = \int_0^x \int_0^y \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu) \left[ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy - \int_0^x \int_0^y pw(x, y) dx dy \quad (3)$$

$$\text{Let } w(x, y) = A_T \varphi(x) \cdot \psi(y) \quad (4)$$

be the solution of equation (3)

Where:

$\varphi(x)$  = Represent x-coordinate

$\psi(y)$  = Represent y-coordinate

$A_T$  = coordinate defining the shape of the deflection surface

Let  $x$  and  $y$  coordinate be represented by the polynomial functions:

$$\varphi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \quad (5)$$

$$\psi(y) = b_0 + b_1y + b_2y^2 + b_3y^3 + b_4y^4 \quad (6)$$

Differentiation of equations (5) and (6) yields:

For  $x$  – direction:

$$\varphi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \quad (7)$$

$$\varphi'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \quad (8)$$

$$\varphi''(x) = 2a_2 + 6a_3x + 12a_4x^2 \quad (9)$$

$$\varphi'''(x) = 6a_3 + 24a_4x \quad (10)$$

Likewise, for  $y$  – direction:

$$\psi(y) = b_0 + b_1y + b_2y^2 + b_3y^3 + b_4y^4 \quad (11)$$

$$\psi'(y) = b_1 + 2b_2y + 3b_3y^2 + 4b_4y^3 \quad (12)$$

$$\psi''(y) = 2b_2 + 6b_3y + 12b_4y^2 \quad (13)$$

$$\psi'''(y) = 6b_3 + 24b_4y \quad (14)$$

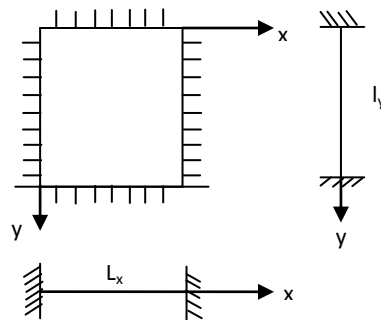


**Figure 1: All round fixed supported plate**

Boundary conditions:

$$\left. \begin{aligned} \varphi(0) = 0 \quad \varphi(l_x) = 0 \quad \varphi\left(\frac{l_x}{2}\right) = 1 \\ \varphi'(0) = 0 \quad \varphi'(l_x) = 0 \end{aligned} \right\} \quad (15)$$

**At  $x = 0$ :**

$$\left. \begin{aligned} \varphi(0) = 0 &= a_0 \\ \varphi'(0) = 0 &= a_1 \end{aligned} \right\} \quad (16)$$

**At  $x = l_x$ :**

$$\left. \begin{aligned} \varphi = 0 &= a_0 + a_1l_x + a_2l_x^2 + a_3l_x^3 + a_4l_x^4 \\ \varphi' = 0 &= a_1 + 2a_2l_x + 3a_3l_x^2 + 4a_4l_x^3 \end{aligned} \right\} \quad (17)$$

At  $x = \frac{l_x}{2}$ :

$$\varphi = 1 = a_0 + \frac{a_1 l_x}{2} + \frac{a_2 l_x^2}{4} + \frac{a_3 l_x^3}{8} + \frac{a_4 l_x^4}{16} \quad (18)$$

Substitution of equations (17) into (18) gives:

$$0 = a_2 l_x^2 + a_3 l_x^3 + a_4 l_x^4 \quad (19)$$

$$a_2 = -l_x (a_3 + a_4 l_x)$$

Substituting equation (19) into (18) gives:

$$0 = 0 - 2l_x^2 (a_3 + a_4 l_x) + 3a_3 l_x^2 + 4a_4 l_x^3 \quad (20)$$

$$a_3 = 2a_4 l_x$$

Substituting equation (17), (19), (20) into (19) gives:

$$1 = 0 + 0 - \frac{l_x^3}{4} (-2a_4 l_x + a_4 l_x) - \frac{2a_4 l_x^4}{8} + \frac{a_4 l_x^4}{16} \quad (21)$$

$$1 = \frac{a_4 l_x^4}{2} - \frac{a_4 l_x^4}{4} - \frac{a_4 l_x^4}{4} + \frac{a_4 l_x^4}{16} = \frac{a_4 l_x^4}{16} \quad (22)$$

$$a_4 = \frac{16}{l_x^4} \quad (23)$$

Substitution of equation (23) into (20) gives:

$$a_3 = \frac{2 \times 16 \times l_x}{l_x^4} = -\frac{32}{l_x^3} \quad (24)$$

Substitution of equation (23) and (24) into (19) gives:

$$a_2 = -l_x \left( \frac{-32}{l_x^3} + \frac{16l_x}{l_x^4} \right) = \frac{16}{l_x^2} \quad (25)$$

Therefore,

$$a_0 = 0, a_1 = 0, a_2 = \frac{16}{l_x^2}, a_3 = \frac{32}{l_x^3}, a_4 = \frac{16}{l_x^4} \quad (26)$$

From equation (26), (5) and (6) we have:

$$\varphi(x) = \frac{16x^2}{l_x^2} - \frac{32x^3}{l_x^3} + \frac{16x^4}{l_x^4} = 16 \left[ \frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right] \quad (27)$$

$$\psi(y) = \frac{16y^2}{l_y^2} - \frac{32y^3}{l_y^3} + \frac{16y^4}{l_y^4} = 16 \left[ \frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right] \quad (28)$$

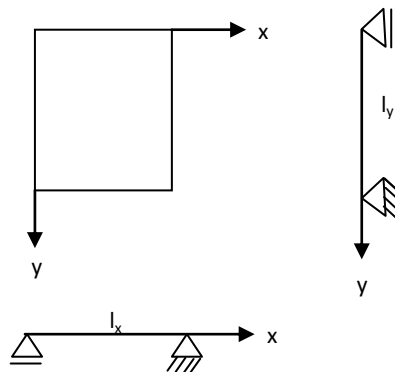


**Figure 2: All round simply supported plate.**

Boundary conditions:

$$\left. \begin{aligned} \varphi_{(0)} &= 0 & \varphi_{(l_x)} &= 0 & \varphi_{\left(\frac{l_x}{2}\right)} &= 1 \\ \varphi''_{(0)} &= 0 & \varphi''_{(l_x)} &= 0 \end{aligned} \right\} \quad (29)$$

At  $x = 0$ :

$$\left. \begin{aligned} \varphi_{(0)} &= 0 = a_0 \\ \varphi''_{(0)} &= 0 = 2a_2 = a_2 \end{aligned} \right\} \quad (30)$$

At  $x = l_x$ :

$$\left. \begin{aligned} \varphi_{(l_x)} &= 0 = a_0 + a_1 l_x + a_2 l_x^2 + a_3 l_x^3 + a_4 l_x^4 \\ \varphi''_{(l_x)} &= 0 = 2a_2 + 6a_3 l_x + 12a_4 l_x^2 \end{aligned} \right\} \quad (31)$$

At  $x = \frac{l_x}{2}$ :

$$\varphi_{\left(\frac{l_x}{2}\right)} = 1 = a_0 + a_1 \frac{l_x}{2} + \frac{a_2 l_x^2}{4} + \frac{a_3 l_x^3}{8} + \frac{a_4 l_x^4}{16} \quad (32)$$

Substitution of equations (30) into (31) yields:

$$\begin{aligned} 0 &= 6a_3 l_x + 12a_4 l_x^2 \\ a_3 &= -2a_4 l_x \end{aligned} \quad (33)$$

Substitution of equations (30) and (31) into (31a) yields:

$$\begin{aligned} 0 &= 0 + a_1 l_x + 0 - 2a_4 l_x^4 + a_4 l_x^4 \\ a_1 &= a_4 l_x^3 \end{aligned} \quad (34)$$

Substitution of equation (30), (33) and (34) into (32) yields:

$$\begin{aligned} 1 &= 0 + \frac{a_4 l_x^4}{2} + 0 - \frac{2a_4 l_x^4}{8} + \frac{a_4 l_x^4}{16} = \frac{5a_4 l_x^4}{16} \\ a_4 &= \frac{16}{5l_x^4} \end{aligned} \quad (35)$$

Substitution of equations (30) into (33) yields:

$$a_3 = \frac{2 \times 16}{5l_x^4} \times l_x = -\frac{32}{5l_x^3} \quad (36)$$

Substitution of equations (35) into (34) yields:

$$a_1 = \frac{16}{5l_x^4} \times l_x^3 = \frac{16}{5l_x} \quad (37)$$

Therefore,

$$a_0 = 0, \quad a_1 = \frac{16}{5l_x}, \quad a_2 = 0, \quad a_3 = -\frac{32}{5l_x^3}, \quad a_4 = \frac{16}{5l_x^4} \quad (38)$$

From equations (38), (5) and (6) we have:

$$\varphi(x) = \frac{16x}{5l_x} - \frac{32x^3}{5l_x^3} + \frac{16x^4}{l_x^4} = \frac{16}{5} \left( \frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \quad (39)$$

$$\psi(y) = \frac{16y}{5l_y} - \frac{32y^3}{5l_y^3} + \frac{16y^4}{l_y^4} = \frac{16}{5} \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (40)$$

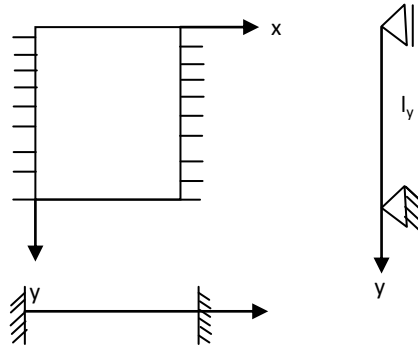


**Figure 3: Two opposite sides fixed and the other two, simply supported plate**

The derived equations for all around fixed and all round simply supported plates are:

$$\phi(x) = \frac{16x^2}{l_x^2} - \frac{32x^3}{l_x^3} + \frac{16x^4}{l_x^4} = 16 \left[ \frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right] \quad (41)$$

$$\psi(y) = \frac{16y}{5l_y} - \frac{32y^3}{5l_y^3} + \frac{16y^4}{5l_y^4} = \frac{16}{5} \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (42)$$

## II. Derivation of Equation for Point Loads

All round fixed rectangular plate:

Substitution equations (27) and (28) into (4) we have:

$$w(x, y) = A.256 \left( \frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \left( \frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (43)$$

Therefore

$$\frac{\partial^2 w}{\partial x^2} = A.256 \left( \frac{2}{l_x^2} - \frac{12x}{l_x^3} + \frac{12x^4}{l_x^4} \right) \left( \frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (44)$$

Similarly,

$$\frac{\partial^2 w}{\partial x^2} = A.256 \left( \frac{2}{l_y^2} - \frac{12y}{l_y^3} + \frac{12y^2}{l_y^4} \right) \left( \frac{x^2}{l_x^2} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \quad (45)$$

Expansion of equations (44) and (45) yields:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} = A.256 & \left( \frac{2y^2}{l_x^2 l_y^2} - \frac{4y^3}{l_x^2 l_y^3} + \frac{2y^4}{l_x^2 l_y^4} - \frac{12xy^2}{l_x^3 l_y^2} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_y^4} + \frac{12x^2 y^2}{l_x^4 l_y^2} \right. \\ & \left. - \frac{24x^2 y^2}{l_x^4 l_y^3} + \frac{12x^2 y^4}{l_x^4 l_y^4} \right) \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{\partial^2 w}{\partial y^2} = A.256 & \left( \frac{2x^2}{l_x^2 l_y^2} - \frac{4x^3}{l_x^3 l_y^2} + \frac{2x^4}{l_x^4 l_y^2} - \frac{12x^2 y^2}{l_x^2 l_y^3} + \frac{24x^3 y}{l_x^3 l_y^3} - \frac{12x^4 y}{l_x^4 l_y^3} + \frac{12x^2 y^2}{l_x^2 l_y^4} \right. \\ & \left. - \frac{24x^3 y^2}{l_x^4 l_y^4} + \frac{12x^4 y^2}{l_x^4 l_y^4} \right) \end{aligned} \quad (47)$$

Therefore,

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \left( A.256 \times \frac{2y^2}{l_x^2 l_y^2} \right) \left( 1 - \frac{2y}{l_y} + \frac{y^2}{l_y^2} - \frac{6x}{l_x} + \frac{12xy}{l_x l_y} - \frac{6xy^2}{l_x l_y^2} + \frac{6x^2}{l_x^2} - \frac{12x^2 y}{l_x^2 l_y} + \frac{6x^2 y^2}{l_x^2 l_y^2} \right) \\ \frac{\partial^2 w}{\partial x^2} &= \left( \frac{A.512 y^2}{l_x^2 l_y^2} \right) \left( 1 - \frac{2y}{l_y} + \frac{y^2}{l_y^2} - \frac{6x}{l_x} + \frac{12xy}{l_x l_y} - \frac{6xy^2}{l_x l_y^2} + \frac{6x^2}{l_x^2} - \frac{12x^2 y}{l_x^2 l_y} + \frac{6x^2 y^2}{l_x^2 l_y^2} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 = A^2 \left( \frac{512y^2}{l_x^2 l_y^2} \right) & \left( 1 - \frac{4y}{l_y} + \frac{6y^2}{l_y^2} - \frac{12x}{l_x} + \frac{48xy}{l_x l_y} - \frac{72xy^2}{l_x l_y^2} + \frac{48x^2}{l_x^2} - \frac{192x^2 y}{l_x^2 l_y} \right. \\ & + \frac{288x^2 y^2}{l_x^2 l_y^2} - \frac{4y^3}{l_y^3} + \frac{48xy^3}{l_x l_y^3} - \frac{192x^2 y^3}{l_x^3 l_y^3} + \frac{y^2}{l_y^4} - \frac{12xy^4}{l_x l_y^4} + \frac{48x^2 y^4}{l_x^2 l_y^4} - \frac{72x^3}{l_x^3} \\ & + \frac{288x^3 y}{l_x^3 l_y} - \frac{432x^3 y^2}{l_x^3 l_y^2} + \frac{288x^3 y^3}{l_x^3 l_y^3} - \frac{72x^3 y^4}{l_x^3 l_y^4} + \frac{36x^4}{l_y^4} - \frac{144x^4 y}{l_x l_y^4} + \frac{216x^4 y^2}{l_x^2 l_y^2} \\ & \left. - \frac{144x^4 y^3}{l_x^4 l_y^3} + \frac{36x^4 y^4}{l_x^4 l_y^4} \right) \end{aligned} \quad (49)$$

Similarly,

$$\begin{aligned} \left(\frac{\partial^2 w}{\partial y^2}\right)^2 = A^2 \left( \frac{512x^2}{l_x^2 l_y^2} \right) & \left( 1 - \frac{4x}{l_x} + \frac{6x^2}{l_x^2} - \frac{12y}{l_y} + \frac{48xy}{l_x l_y} - \frac{72x^2 y}{l_x^2 l_y} + \frac{48y^2}{l_y^2} - \frac{192xy^2}{l_x l_y^2} \right. \\ & + \frac{288x^2 y^2}{l_x^2 l_y^2} - \frac{4x^3}{l_x^3} + \frac{48x^3 y}{l_x^3 l_y} - \frac{192x^3 y^2}{l_x^3 l_y^2} + \frac{x^4}{l_x^4} + \frac{12x^4 y}{l_x^4 l_y} + \frac{48x^4 y^2}{l_x^4 l_y^2} - \frac{72y^3}{l_y^3} \\ & + \frac{288xy^3}{l_x l_y^3} - \frac{432x^2 y^3}{l_x^2 l_y^3} + \frac{288x^3 y^3}{l_x^3 l_y^3} - \frac{72x^4 y^3}{l_x^4 l_y^3} + \frac{36y^4}{l_y^4} - \frac{144xy^4}{l_x l_y^4} + \frac{216x^2 y^4}{l_x^2 l_y^4} \\ & \left. - \frac{144x^3 y^4}{l_x^3 l_y^4} + \frac{36x^4 y^4}{l_x^4 l_y^4} \right) \end{aligned} \quad (50)$$

The strain energy equation for thin rectangular isotropic plate is given by:

$$U = \int_0^{l_x} \int_0^{l_y} \frac{D}{2} \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu) \left[ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy$$

According to Gould [5], for plates that have a polygonal plane form and straight edges, the third term in the energy equation is neglected. The reduced form of equation (50) is now.

$$U = \int_0^{l_x} \int_0^{l_y} \frac{D}{2} \left( \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right) dx dy \quad (51)$$

Substitution of the derivatives of equations (49) and (50) into equation (51) and integrating rigorously yields:

$$U = \frac{D}{2} \cdot A^2 \cdot \left( \frac{512}{l_x^2 l_y^2} \right)^2 \left( \frac{7}{22050} (l_y^4 + l_x^4) \right) \quad (52)$$

Substituting equation (43) into (1), the potential energy due to the external load, it becomes:

$$V = -P.A. \frac{256\zeta^2 \eta^2}{l_x^2 l_y^2} \beta_1$$

where  $\beta_1 = 1 - \frac{2\eta}{l_y} + \frac{\eta^2}{l_y^2} - \frac{2\zeta}{l_x} + \frac{4\zeta\eta}{l_x l_y} + \frac{2\zeta\eta}{l_x l_y^2} + \frac{\zeta^2}{l_x^2} - \frac{2\zeta^2 \eta}{l_x^2 l_y} + \frac{\zeta^2 \eta^2}{l_x^2 l_y^2}$  (53)

The total potential energy in the plate becomes

$$\begin{aligned} \Pi_T &= U + (-V) \\ \Pi_T &= U - V \end{aligned} \quad (54)$$

Substitution of equations (52) and (53) into (54) and making the total potential energy a minimum, we have:

$$0 = \frac{D}{2} \cdot A^2 \cdot \left( \frac{512}{l_x^2 l_y^2} \right)^2 \frac{7 l_x l_y}{22050} (l_x^4 + l_y^4) - \frac{P.A.256\zeta^2 \eta^2}{l_x^2 l_y^2} \beta_1 \quad (55)$$

Using Rayleigh- Ritz method, the coefficient A is obtained by taking the variation of  $\Pi$  with respect to A and equating it to zero.

$$\frac{\partial \Pi_T}{\partial A} = 0$$

$$\Rightarrow A = \frac{P \zeta^2 \eta^2 l_x l_y 0.32507965}{D(l_y^4 + l_x^4)} \beta_1 \quad (56)$$

Substitution of equation (56) into (4) yields:

$$w(x, y) = \frac{P \zeta^2 \eta^2 l_x l_y 0.32507965 \beta_1}{D(l_y^4 + l_x^4)} \varphi_{(x)} \psi_{(y)} \quad (57)$$

### III. All round simply supported plate

Substitution of equations (39) and (40) into (4) gives:

$$w(x, y) = A \frac{16}{5} \left( \frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \cdot \frac{16}{5} \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right)$$

$$= A \frac{256}{25} \left( \frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \cdot \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (58)$$

Therefore,

$$\frac{\partial w}{\partial x} = A \frac{256}{25} \left( \frac{1}{l_x} - \frac{6x^2}{l_x^3} + \frac{4x^3}{l_x^4} \right) \cdot \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right)$$

$$\frac{\partial^2 w}{\partial x^2} = A \frac{256}{25} \left( -\frac{12x}{l_x^3} + \frac{12x^2}{l_x^4} \right) \cdot \left( \frac{y}{l_y} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4} \right) \quad (59)$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = A \frac{256}{25} \left( -\frac{12y}{l_y^3} + \frac{12y^2}{l_y^4} \right) \cdot \left( \frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4} \right) \quad (60)$$

Expansion of equation (59) yields:

$$\frac{\partial^2 w}{\partial x^2} = A \frac{256}{25} \left( \frac{-12xy}{l_y^3 l_y} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_x^4} + \frac{12x^2 y}{l_x^3 l_x} - \frac{24x^2 y^3}{l_x^4 l_y^3} + \frac{12x^2 y^4}{l_x^4 l_y^4} \right)$$

$$\frac{\partial^2 w}{\partial x^2} = A \frac{3072xy}{25 l_x l_y} \left( -\frac{1}{l_y^2} + \frac{2y^2}{l_x^2 l_y^2} - \frac{y^3}{l_x^2 l_y^3} + \frac{x}{l_x^3} - \frac{2xy^2}{l_x^3 l_y^2} + \frac{xy^3}{l_x^3 l_y^3} \right) \quad (61)$$

$$\Rightarrow \left( \frac{\partial^2 w}{\partial x^2} \right)^2 = A^2 \left( \frac{3072xy}{25 l_x l_y} \right)^2 \left( \frac{1}{l_x^4} - \frac{4y^2}{l_x^4 l_y^2} + \frac{2y^3}{l_x^4 l_y^3} - \frac{2x}{l_x^5} + \frac{8xy^2}{l_x^5 l_y^2} - \frac{4xy^3}{l_x^5 l_y^3} + \frac{4y^4}{l_x^4 l_y^4} - \frac{4y^5}{l_x^4 l_y^5} \right.$$

$$- \frac{8xy^4}{l_x^5 l_y^4} + \frac{8xy^5}{l_x^5 l_y^5} + \frac{y^6}{l_x^4 l_y^6} - \frac{2xy^6}{l_x^5 l_y^6} + \frac{x^2}{l_x^6} + \frac{4x^2 y^2}{l_x^6 l_y^2} + \frac{2x^2 y^3}{l_x^6 l_y^3} + \frac{4x^2 y^4}{l_x^6 l_y^4}$$

$$\left. - \frac{4x^2 y^5}{l_x^6 l_y^5} + \frac{x^2 y^6}{l_x^6 l_y^6} \right) \quad (62)$$

Similarly,

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^2 = A^2 \left(\frac{3072xy}{25l_x l_y}\right)^2 \left(\frac{1}{l_y^4} - \frac{4x^2}{l_x^2 l_y^4} + \frac{2x^3}{l_x^3 l_y^4} - \frac{2y}{l_y^5} + \frac{8x^2 y}{l_x^2 l_y^5} - \frac{4x^3 y}{l_x^3 l_y^5} + \frac{4x^4}{l_x^4 l_y^4} - \frac{4x^5}{l_x^5 l_y^4} - \frac{8x^4 y}{l_x^4 l_y^5} + \frac{8x^5 y}{l_x^5 l_y^5} + \frac{x^6}{l_x^4 l_y^6} - \frac{2x^6 y}{l_x^5 l_y^6} + \frac{x^2}{l_x^6} + \frac{4x^2 y^2}{l_x^2 l_y^6} + \frac{2x^3 y^2}{l_x^3 l_y^6} + \frac{4x^4 y^2}{l_x^4 l_y^6} - \frac{4x^5 y^2}{l_x^5 l_y^6} + \frac{x^6 y^2}{l_x^6 l_y^6}\right) \quad (63)$$

Substituting the derivatives of equations (62) and (63) into (51) and integrating, yields:

$$U = \frac{DA^2}{2} \left(\frac{3072}{25l_x l_y}\right)^2 \left[\frac{31}{18900} \left(\frac{l_y^3}{l_x} + \frac{l_x^3}{l_y}\right)\right] = \frac{DA^2 12.383184}{l_x^2 l_y^2} \left(\frac{l_y^3}{l_x} + \frac{l_x^3}{l_y}\right) \quad (64)$$

#### IV. Concentrated Load

Upon substitution of equation (58) into (1) the potential energy due to the external load becomes:

$$V = -\frac{P.A.256\zeta\eta}{25l_x l_y} \beta_2 \quad (65)$$

Where:

$$\beta_2 = 1 - \frac{2\eta^2}{l_y^2} + \frac{\eta^3}{l_y^3} - \frac{2\zeta^2}{l_x^2} + \frac{4\zeta\eta^2}{l_x^2 l_y^2} - \frac{2\zeta^2 \eta^2}{l_x^3 l_y^3} + \frac{\zeta^3}{l_x^3} - \frac{2\zeta^3 \eta^2}{l_x^3 l_y^2} + \frac{\zeta^3 \eta^3}{l_x^3 l_y^3}$$

Substitution of equations (64) and (65) into (54) making equation it a minimum yields:

$$A = \frac{P256\zeta\eta l_x l_y}{D24.766366 \left[\frac{l_y^3}{l_x} + \frac{l_x^3}{l_y}\right] \times 25} \beta_2 \quad (66)$$

Substitution of equation (66) into (4) yields:

$$w(x, y) = \frac{0.413463969 P \zeta \eta l_x l_y}{D \left[\frac{l_y^3}{l_x} + \frac{31 l_x^3}{6 l_y}\right]} \beta_2 \varphi(x) \psi(y) \quad (67)$$

#### V. Two opposite sides fixed and the other side simply supported

Substitution of equations (41) and (42) into (4) yields:

$$\therefore w(x, y) = A \frac{16}{5} \left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) 16 \left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \quad (68)$$

$$w(x, y) = A \frac{256}{5} \left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = A \frac{256}{5} \left(-\frac{12x}{l_x^3} + \frac{12x^2}{l_x^4}\right) \left(\frac{y^2}{l_y^2} - \frac{2y^3}{l_y^3} + \frac{y^4}{l_y^4}\right) \quad (69)$$

$$\frac{\partial^2 w}{\partial y^2} = A \frac{256}{5} \left(\frac{2}{l_y^2} - \frac{12y}{l_y^3} + \frac{12y^2}{l_y^4}\right) \left(\frac{x}{l_x} - \frac{2x^3}{l_x^3} + \frac{x^4}{l_x^4}\right) \quad (70)$$

Expansion of equations (69) and (70) yields:

$$\frac{\partial^2 w}{\partial x^2} = A \frac{256}{5} \left(\frac{12xy^2}{l_x^3 l_y^2} + \frac{24xy^3}{l_x^3 l_y^3} - \frac{12xy^4}{l_x^3 l_y^4} + \frac{12x^2 y^2}{l_x^4 l_y^2} - \frac{24x^2 y^3}{l_x^4 l_y^3} + \frac{12x^2 y^4}{l_x^4 l_y^4}\right) \quad (71)$$

$$\frac{\partial^2 w}{\partial y^2} = A \frac{256}{5} \left( \frac{2x}{l_x l_y^2} - \frac{4x^3}{l_x^3 l_y^2} - \frac{2x^4}{l_x^4 l_y^2} + \frac{12xy}{l_x^3 l_y^3} + \frac{24x^3 y}{l_x^3 l_y^3} - \frac{12x^4 y}{l_x^4 l_y^3} + \frac{12xy^2}{l_x^3 l_y^4} - \frac{24x^3 y^2}{l_x^3 l_y^4} + \frac{12x^4 y^2}{l_x^4 l_y^4} \right) \quad (72)$$

Therefore,

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= A \frac{256}{5} \cdot \frac{2xy^2}{l_x^3 l_y^2} \left( -6 + \frac{12y}{l_y} - \frac{6y^4}{l_y^2} + \frac{6x}{l_x} - \frac{12xy}{l_x l_y} + \frac{6xy^2}{l_x l_y^2} \right) \\ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 &= A^2 \left( \frac{512xy^2}{5l_x^3 l_y^2} \right)^2 \left( 36 - \frac{144y}{l_y} + \frac{216y^2}{l_y^2} - \frac{72x}{l_x} + \frac{288xy}{l_x l_y} - \frac{432xy^2}{l_x l_y^2} - \frac{144y^3}{l_y^3} + \frac{288xy^3}{l_x l_y^3} \right. \\ &\quad \left. + \frac{36y^4}{l_y^4} - \frac{72xy^4}{l_x l_y^4} + \frac{36x^2}{l_y^2} - \frac{144x^2 y}{l_y^2 l_y} + \frac{216x^2 y^2}{l_x^2 l_y^2} - \frac{144x^2 y^3}{l_y^2 l_y^3} + \frac{36x^2 y^4}{l_y^2 l_y^4} \right) \end{aligned} \quad (73)$$

Using equation (72), we have:

$$\begin{aligned} \left( \frac{\partial^2 w}{\partial x^2} \right) &= A \frac{256 \times 2x}{5l_x l_y^2} \left( 1 - \frac{2x^2}{l_x^2} + \frac{x^3}{l_x^3} - \frac{6y}{l_y} + \frac{12x^2 y}{l_x^2 l_y} - \frac{6x^3 y}{l_x^3 l_y} + \frac{6y^2}{l_y^2} - \frac{12x^2 y^2}{l_x^2 l_y^2} + \frac{6x^3 y^2}{l_x^3 l_y^2} \right) \\ \left( \frac{\partial^2 w}{\partial y^2} \right) &= A^2 \left( \frac{512x}{5l_x l_y^2} \right)^2 \left( 1 - \frac{4x^2}{l_x^2} + \frac{2x^3}{l_x^3} - \frac{12y}{l_y} + \frac{48x^2 y}{l_x^2 l_y} - \frac{24x^3 y}{l_x^3 l_y} + \frac{48y^2}{l_y^2} - \frac{192x^2 y^2}{l_x^2 l_y^2} \right. \\ &\quad + \frac{96x^3 y^2}{l_x^3 l_y^2} + \frac{4x^4}{l_x^4} - \frac{4x^5}{l_x^5} - \frac{48x^4 y}{l_x^4 l_y} + \frac{48x^5 y}{l_x^5 l_y} + \frac{192x^4 y^2}{l_x^4 l_y^2} - \frac{192x^5 y^2}{l_x^5 l_y^2} + \frac{x^6}{l_x^6} \\ &\quad - \frac{12x^6 y}{l_x^6 l_y} + \frac{48x^6 y^2}{l_x^6 l_y^2} - \frac{72y^3}{l_y^3} + \frac{288x^2 y^3}{l_x^2 l_y^3} - \frac{144x^3 y^3}{l_x^3 l_y^3} - \frac{288x^4 y^3}{l_x^4 l_y^3} + \frac{288x^5 y^3}{l_x^5 l_y^3} - \frac{72x^6 y^3}{l_x^6 l_y^3} \\ &\quad \left. + \frac{36y^4}{l_y^4} - \frac{144x^2 y^4}{l_x^2 l_y^4} + \frac{72x^3 y^4}{l_x^3 l_y^4} + \frac{144x^4 y^4}{l_x^4 l_y^4} - \frac{144x^5 y^4}{l_x^5 l_y^4} + \frac{36x^6 y^4}{l_x^6 l_y^4} \right) \end{aligned} \quad (74)$$

Substitution of equations (73) and (74) into (51) and integrating yields:

$$U = \frac{D}{2} A^2 \left( \frac{512}{5l_x l_y^2} \right)^2 \left\{ \frac{1}{525} \left( \frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y} \right) \right\} \quad (75)$$

Substitution of equation (68) into (1) gives the potential energy due to the external load as:

$$V = - \frac{P \cdot A \cdot 256 \xi \eta^2}{5l_x l_y^2} \beta_3 \quad (76)$$

Where:

$$\beta_3 = 1 - \frac{2\eta}{l_y} + \frac{\eta^2}{l_y^2} - \frac{2\xi}{l_x} + \frac{4\xi\eta}{l_x l_y} - \frac{2\xi\eta^2}{l_x l_y^2} + \frac{\xi^2}{l_x^2} - \frac{2\xi^2\eta}{l_x^2 l_y} + \frac{\xi^2\eta^2}{l_x^2 l_y^2}$$

Substitution of equations (75) and (76) into (54) and making it a minimum yields:

$$A = \frac{0.004882812 P \xi \eta^2 l_x \beta_3}{D \left\{ \frac{1}{525} \left( \frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y} \right) \right\}} \quad (77)$$

Substitution of equation (77) into (4) yields:

$$w(x, y) = \frac{0.004882812 P \xi \eta^2 l_x \beta_3}{D \left\{ \frac{1}{525} \left( \frac{l_y^3}{l_x} + \frac{31l_x^3}{6l_y} \right) \right\}} \varphi(x) \psi(y) \quad (78)$$

## VI. Results and Discussion

### An example for numerical study

A numerical example is given to demonstrate the applicability of energy approach in the determination of numerical factors for deflection of all round fixed, all round simply supported and two opposite sides fixed and the other two opposite side simply supported a centre-point loading. Maximum deflection of the plate is at the

$$\text{middle } \xi = x = \frac{l_x}{2}, \eta = y = \frac{l_y}{2}.$$

**Table 1: Comparison of results of numerical factors for deflection of a centrally loaded isotropic plate for varying support conditions at varying aspect ratio  $l_y/l_x$ .**

| $l_y/l_x$ | Support condition and numerical factors for deflection |                          |                           |
|-----------|--------------------------------------------------------|--------------------------|---------------------------|
|           | ARF                                                    | ASS                      | 2opp (F&SS)               |
| 1.0       | $\frac{0.010158718P}{D}$                               | $\frac{0.090445243P}{D}$ | $\frac{0.051962357P}{D}$  |
| 1.1       | $\frac{0.010974598P}{D}$                               | $\frac{0.097709199P}{D}$ | $\frac{0.0643211243P}{D}$ |
| 1.2       | $\frac{0.011422609P}{D}$                               | $\frac{0.10169793P}{D}$  | $\frac{0.076476586P}{D}$  |
| 1.3       | $\frac{0.011575791P}{D}$                               | $\frac{0.103061746P}{D}$ | $\frac{0.087749614P}{D}$  |
| 1.4       | $\frac{0.011515005P}{D}$                               | $\frac{0.10252055P}{D}$  | $\frac{0.195214552P}{D}$  |
| 1.5       | $\frac{0.011310738P}{D}$                               | $\frac{0.10070192P}{D}$  | $\frac{0.105723818P}{D}$  |

## VII. Discussion of Results and Conclusion

The results of numerical factors for deflection of a thin rectangular isotropic plate of three varying edge conditions subjected to a concentrated point load is as shown in Table 1. From Table 1, it can be seen that the numerical factors for deflection decrease as the aspect ratio of the plate increases up to an aspect ratio of 1.3 after which there was a decrease in the value of the numerical factors (1.4 -1.5). Also from Table 1, the deflection factors have higher values in the case of all round simply supported plate followed by two opposite sides fixed and the other simply supported plate while smaller values are observed for all round fixed plates.

In conclusion, numerical factors for deflection of a centrally loaded thin rectangular plate subjected to different support conditions is a function of the assumed polynomial function. The numerical factors for deflection decreased with increase in the plate aspect ratio up to a value of 1.3. The optimum value of aspect ratio for the three support conditions for economic design of plate was 1.3. The all round simply supported plates have the highest values of numerical factors for deflection followed by two opposite sides simply supported and the other sides fixed and then all around fixed plates.

## VIII. References

- [1] Biot, M.A. "A New Approach to the Mechanics of Orthotropic Multilayered Plates." Int. J. Solid Structure, 1972, Vol.9, pp.613-490 Pergamon Press, Great Britain.
- [2] Charlton, T.M., Energy Principles in Applied Statics. Blackie, London, 1959.
- [3] Charlton, T.M., Analysis of Statically indeterminate framework, Longmans, London, 1961, pp. 102.
- [4] Coull, A. and Dykes, A.R., Fundamentals of Structural Theory. McGraw-Hill, Maidenhead, 1972.
- [5] Gould, P.L., Analysis of Shells and Plates. Prentice-Hall, Upper saddle River, New Jersey, 1999.
- [6] Chajes, A., Principles of structural stability theory. Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1974.
- [7] El Naschie, M.S., Stress, stability and Chaos in structural Engineering: An Energy Approach. McGraw-Hill Book Company, London, 1990.