



COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACES

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Abstract: In this paper, we have common fixed point theorem has generalization of result [6] and the condition for continuous self mapping S, T, U of complete fuzzy metric space C, X, M have a unique common fixed point in X .

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I. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [11]. Since then, it was developed extensively by many authors and used in various field. George and Veeramani modified the concept of fuzzy metric space which introduced by Kramosil and Michalek. The concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric space in the sense of George and Veeramani with continuous T -norm* defined by

$$a * b \leq \min\{a, b\} \text{ for all } a, b \in [0, 1]$$

In this paper we have a generalization of the result obtained in [6]. We characterize the condition for three continuous self mapping of complete fuzzy metric space have a unique common fixed point.

II. Preliminaries

A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if $([0, 1], *)$ is an abelian topological monoid with such that $a * b \leq c * d$ whenever $a \leq c, b \leq d$ for all $a, b, c, d \in [0, 1]$.

The 3 tuple $(X, M, *)$ is called a fuzzy metric space [11] if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfy the following condition.

- (1) $M(x, y, t) > 0$
- (2) $M(x, y, t) = 1$, if and only if $x = y$
- (3) $M(x, y, t) = M(y, x, t)$
- (4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- (5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous, for all $x, y, z \in X$ and $t, s > 0$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called cauchy sequence if for every $\epsilon > 0$ and each $t > 0$ there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+p}, t) > 1 - \epsilon$ for all $n \geq n_0$ and $t > 0$.

A fuzzy metric space in which every cauchy sequence is convergent is said to be complete. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called cauchy sequence if for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$

Self mapping A and B of fuzzy metric space $(X, M, *)$ is said to be compatible [4] if $\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1$.

III. Common Fixed Point Theorem

Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, C, S, T, U be mappings from X in to it self such that the condition are satisfied.

- (i) $AX \subset TX, BX \subset SX, BX \subset UX, CX \subset TX$
- (ii) S, T and U are continuous.
- (iii) The pair [A, S], [B, T] and [C, U] are compatible.
- (iv) There exist $q \in (0, 1)$ such that every $x, y, z \in X$ and $t > 0$

$$M(Ax, By, Cz, qt) \geq M(Sx, Ty, Uz, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Cz, Uz, t) \\ * M(Ax, Ty, t) * M(By, Uz, t)$$

then A, B, C, S, T and U have a unique common fixed point in X.

Proof.

Let $x_0 \in X$ from (i) there exist $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$ from (i) there exist $x_2 \in X$ Such that $Bx_1 = Sx_2$

and for this $x_2 \in X$ from (i) there exist $x_3 \in X$ such that $Bx_2 = Ux_3$

and for this $x_3 \in X$ from (i) there exists $x_4 \in X$ such that $Cx_3 = Tx_4$

Inductively we can find a sequence $\{y_n\}$ in X as follows

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$$

$$y_{2n} = Sx_{2n} = Bx_{2n-1}$$

$$y_{2n+1} = Ux_{2n+1} = Bx_{3n-1}$$

$$y_{2n+2} = Tx_{2n+2} = Cx_{2n+1}, \text{ for } n = 1, 2, 3, \dots$$

from (iv) we have

$$M\{y_{2n+1}, y_{2n+2}, y_{2n+2}, qt\} = M(Ax_{2n}, Bx_{2n+1}, Cx_{2n+1}, qt) \\ \geq M(Sx_{2n}, Tx_{2n+1}, Ux_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) \\ * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Cx_{2n+1}, Ux_{2n+1}, t) \\ * M(Ax_{2n}, Tx_{2n+1}, t) * M(Bx_{2n+1}, Ux_{2n+1}, t)$$

$$= M(y_{2n}, y_{2n+1}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) \\ * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t)$$

$$\geq M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t)$$

we have

$$M(y_{2n+1}, y_{2n+2}, y_{2n+2}, qt) \geq M(y_{2n+1}, y_{2n}, t)$$

Similarly we have also

$$M(y_{2n+2}, y_{2n+3}, y_{2n+3}, qt) \geq M(y_{2n+2}, y_{2n+1}, t)$$

Thus we have (generalize)

$$M(y_{n+1}, y_{n+2}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t) \text{ for } n = 1, 2, 3, \dots \text{ and so on Then}$$

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n+1}, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \dots \geq M(y_1, y_2, t/q^n) \rightarrow 1$$

as $n \rightarrow \infty$ and hence $M(y_n, y_{n+1}, t) \rightarrow 1$ as $n \rightarrow \infty$ for any $t > 0$ for each $\epsilon > 0$ and each $t > 0$ we can choose $n_0 \in \mathbb{N}$ such that $M(y_n, y_{n+1}, t) > 1 - \epsilon$ for all $n > n_0$ for $m, n \in \mathbb{N}$ we suppose $m \geq n$ then we have

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) \dots * M(y_{m-1}, y_m, t/m-n)$$

$$> (m-n) / (1-\epsilon) \dots (1-\epsilon)$$

$$\geq (1-\epsilon)$$

and hence y_n is a cauchy sequence in X since $\{X, M, *\}$ is complete y_n converges to some point $z \in X$ and so

$$\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}, \{Tx_{2n-1}\}, \{Bx_{3n-1}\}, \{Tx_{2n+2}\}, \{Ux_{2n+1}\}, \{Cx_{2n+1}\}$$

also converges to z then

$$ASx_{2n} \rightarrow Sz \dots \dots \dots (1.1)$$

$$BTx_{2n-1} \rightarrow Tz \dots \dots \dots (1.2)$$

$$CUx_{2n+1} \rightarrow Uz \dots \dots \dots (1.3)$$

from (iv)

$$M(ASx_n, BTx_{2n-1}, qt) \geq M(SSx_{2n}, TTx_{2n-1}, t) * M(ASx_{2n}, SSx_{2n}, t) * M(BTx_{2n-1}, TTx_{2n-1}, t) * M(ASx_{2n}, TTx_{2n-1}, t)$$

Taking $n \rightarrow \infty$ using (1.1) (1.2)

$$M(Sz, Tz, qt) \geq M(Sz, Tz, t) * M(Sz, Sz, t) * M(Tz, Tz, t) * M(Sz, Tz, t)$$

$$\geq M(Sz, Tz, t) * M(Sz, Tz, t)$$

$$\geq M(Sz, Tz, t)$$

$$\text{and hence } Sz = Tz \dots \dots \dots (1.4)$$

Again from (iv) and [6]

$$M(ASx_{2n}, CUx_{2n+1}, qt) \geq M(SSx_{2n}, UUx_{2n+1}, t) * M(ASx_{2n}, SSx_{2n}, t) * M(CUx_{2n+1}, UUx_{2n+1}, t) * M(ASx_{2n}, UUx_{2n+1}, t)$$

Taking $n \rightarrow \infty$ ad using (1.1)(1.2)(1.3)

$$M(Sz, Uz, qt) \geq M(Sz, Uz, t) * M(Sz, Sz, t) * M(Uz, Uz, t) * M(Sz, Uz, t)$$

$$\begin{aligned} &\geq M(Sz, Uz, t) * M(Sz, Uz, t) \\ &\geq M(Sz, Uz, t) \\ &\Rightarrow Sz = Uz \end{aligned} \dots\dots\dots(1.5)$$

Now again from (iv)

$$M(Az, BTx_{2n-1}, qt) \geq M(Sz, TTx_{2n-1}, t) * M(Az, Sz, t) * M(BTx_{2n-1}, TTx_{2n-1}, t) * M(Az, TTx_{2n-1}, t)$$

Which implies that taking limit as $n \rightarrow \infty$ and using (1.2) (1.3) (1.4)

$$\begin{aligned} M(Az, Tz, qt) &\geq M(Sz, Sz, t) * M(Az, Tz, t) * M(Tz, Tz, t) * M(Az, Tz, t) \\ &\geq M(Az, Tz, t) \end{aligned}$$

$$\text{and hence } Az = Tz \dots\dots\dots(1.6)$$

From (v), and (1.4) (1.5.) (1.6)

$$\begin{aligned} M(Az, Bz, qt) &\geq M(Sz, Tz, t) * M(Az, Sz, t) * M(Bz, Tz, t) * M(Az, Tz, t) \\ &= M(Az, Az, t) * M(Az, Az, t) * M(Bz, Bz, t) * M(Az, Az, t) \\ &\geq M(Az, Bz, t) \\ &\Rightarrow Az = Bz \end{aligned} \dots\dots\dots(1.7)$$

Again from (iv) and (1.4) to (1.7)

$$\begin{aligned} M(Az, Cz, qt) &\geq M(Sz, Tz, t) * M(Az, Sz, t) * M(Cz, Tz, t) * M(Az, Tz, t) \\ &= M(Az, Az, t) * M(Az, Az, t) * M(Cz, Az, t) * M(Az, Az, t) \\ &\geq M(Az, Cz, t) \end{aligned}$$

$$\text{and so } Az = Cz \dots\dots\dots(1.8)$$

from (1.4) (1.5) (1.6)(1.7)(1.8)

$$\begin{aligned} Az = Bz = Cz = Sz = Tz = Uz \\ \dots\dots\dots(1.9) \end{aligned}$$

now we show $Cz = z$

from (iv)

$$M(Ax_{2n}, Cz, qt) \geq M(Sx_{2n}, Tz, T) * M(Ax_{2n}, Sx_{2n}, t) * M(Cz, Tz, t) * M(Ax_{2n}, Tz, t)$$

which implies that

as taking $n \rightarrow \infty$ and using (1.9)

$$\begin{aligned} M(Z, Cz, qt) &\geq M(Z, Tz, t) * M(Z, Z, t) * M(Cz, Tz, t) * M(Z, Tz, t) \\ &\geq M(Z, Cz, t) * M(Cz, Cz, t) * M(Z, Cz, t) \\ &\geq M(Z, Cz, t) \end{aligned}$$

and hence $Cz = z$

Similarly,

$$M(Ax_{2n}, Uz, qt) \geq M(Sx_{2n}, Tz, t) * M(Ax_{2n}, Sx_{2n}, t) * M(Uz, Tz, t) * M(Ax_{2n}, Tz, t)$$

which implies that as taking $n \rightarrow \infty$ and using (1.9)

$$\begin{aligned} M(z, Uz, qt) &\geq M(z, Tz, t) * M(z, z, t) * M(Uz, Tz, t) * M(z, Tz, t) \\ &\geq M(z, Uz, t) * M(Uz, Uz, t) * M(z, Uz, t) \\ &\geq M(z, Uz, t) \end{aligned}$$

and hence $Uz = z$

and hence $Cz = z = Uz$ thus from (1.9)

z is a common fixed point at A, B, C, S, T and U .

For uniqueness

Let z_1 be another common fixed point of A, B, C, S, T, U then

$$\begin{aligned} M(z, z_1, qt) &= M(Az, Bz_1, qt) \\ &\geq M(Sz, Tz_1, t) * M(Az, Sz, t) * M(Bz_1, Tz_1, t) * M(Az, Tz_1, t) \\ &\geq M(z, z_1, t), \end{aligned}$$

further $z = z_1$ This is proof of the theorem.

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