



Critical Graphs in K - tuple Domination

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Abstract: In this paper we study the concept of edge critical graph w. r. t K - tuple Domination. We have derived characterization of edges which are responsible to increase or decrease K - tuple Domination Number of a graph. We have also given examples to illustrate the conditions given in theorems.

Keywords: K - tuple Dominating Set, minimal K - tuple Dominating Set, minimum K - tuple Dominating Set, K - tuple Domination Number.

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I. Introduction:

Suppose G is a graph. S is a $\mathcal{V}$ set. If we remove an edge from graph or if we add an edge between two non-adjacent vertices then domination number increase, decrease or remains unchanged. This topic has been studied by several authors. Reader may refer to references. We shall prove results regarding this phenomenon for K - tuple Dominating Set.

Definitions 1: K – tuple dominating set: Let G be a graph and K be an integer, $K \geq 2$. A subset S of $V(G)$ is said to be a K – tuple dominating set if following conditions satisfied.

- (1) If $v \in S$ then v is adjacent to at least $K - 1$ vertices of S .
- (2) If $v \notin S$ then v is adjacent to at least K vertices of S .

Definitions 2: Minimal K – tuple dominating set: A K – tuple dominating set S is said to be a minimal K – tuple dominating set if no proper subset $S' \subset S$ is a K – tuple dominating set.

Definitions 3: Minimum K – tuple dominating set: A minimal K – tuple dominating set with minimum cardinality is called minimum K – tuple dominating set.

Definitions 4: K – tuple domination number: The K – tuple domination number γ_{KT} of a graph G equals the minimum cardinality of a K – tuple dominating set in G .

Note: It is useful to partition the Edges of G into two sets according to how their removal affects K – tuple domination number.

Let $E = E_{KT}^+ \cup E_{KT}^0$ for

$$E_{KT}^+ = \{e \in E(G) / \gamma_{KT}(G - e) > \gamma_{KT}(G)\}$$

$$E_{KT}^0 = \{e \in E(G) / \gamma_{KT}(G - e) = \gamma_{KT}(G)\}$$

II. ASSUMPTION:

- (1) In this paper, we have assumed that K is any integer and $K \geq 1$.
- (2) All graphs considered are simple.
- (3) If u and v are end vertices of any edge e then e will also be denoted by $e = uv$
- (4) We consider only those graphs for which the minimum degree is at least $K+1$. So when an edge is removed, the resulting graph has minimum degree at least K .

Note: First we consider the case of removing an edge from given graph. It may be noted that in this case K – tuple domination number of graph will not decrease.

Lemma 5: Let G be a graph and $e = uv$ be an edge of graph G then $\gamma_{KT}(G - e) \geq \gamma_{KT}(G)$.

Proof: Let G be a graph and $e = uv$ be an edge of graph G .

Suppose $\gamma_{KT}(G-e) < \gamma_{KT}(G)$. Let T be γ_{KT} set of $G-\{e\}$. then $|T| < \gamma_{KT}(G)$

There are four possibilities for γ_{KT} set T of $G-\{e\}$.

- (1) $u \in T, v \in T$ (2) $u \notin T, v \notin T$ (3) $u \in T, v \notin T$ (4) $u \notin T, v \in T$

In all cases, T is a K -tuple dominating set in graph G also and $|T| < \gamma_{KT}(G)$.

which is a contradiction. So $\gamma_{KT}(G-e) < \gamma_{KT}(G)$ is not possible.

So, $\gamma_{KT}(G-e) \geq \gamma_{KT}(G)$

Theorem 6: An edge $e=uv \in E_{KT}^+$ $\Leftrightarrow$ (for every γ_{KT} set S of graph G)

- (1) $u \in S, v \notin S$ and v is adjacent to exactly K vertices of S including u .
 (2) $u \in S, v \in S$ then v is adjacent to exactly $K-1$ vertices of S including u or u is adjacent to exactly $K-1$ vertices of S including v .

Proof: $\Leftarrow$ (Sufficient)

Let S be a γ_{KT} set of the graph G and $e=uv$. Suppose one of these two conditions holds.

Now if we remove an edge $e=uv$ then S will not be a K -tuple dominating set in graph $G-\{e\}$. This is true for every γ_{KT} set S of graph G . so, by lemma 5, $\gamma_{KT}(G-e) \geq \gamma_{KT}(G)$.

Suppose $\gamma_{KT}(G-e) = \gamma_{KT}(G)$. Let T be γ_{KT} set in graph $G-\{e\}$ with $|T| = |S|$.

Case 1: $u \in T, v \notin T$. By definition of K -tuple dominating set v will be adjacent to at least K vertices of T of $G-\{e\}$. Consider the graph G , T will be K -tuple dominating set in graph G also and $|T| = |S|$.

So, T will be γ_{KT} set of graph G also. Moreover $u \in T, v \notin T$ and v is now adjacent to at least $K+1$ vertices of T including u . which contradicts condition (1).

Case 2: $u \in T, v \in T$. By definition of K -tuple dominating set v will be adjacent to at least $K-1$ vertices of T and u will be adjacent to at least $K-1$ vertices of T . Consider the graph G , T will be K -tuple dominating set in graph G also and $|T| = |S|$. So, T will be γ_{KT} set of graph G . Moreover $u \in T, v \in T$ and v is now adjacent to at least K vertices of T including u in G and u is now adjacent to at least K vertices of T including v in G . which contradicts condition (2).

So, $\gamma_{KT}(G-e) = \gamma_{KT}(G)$ is not possible. So, $\gamma_{KT}(G-e) > \gamma_{KT}(G)$ and $e=uv \in E_{KT}^+$.

$\Rightarrow$ (Necessity)

Let $e=uv \in E_{KT}^+$ means removal of edge $e=uv$ increase K -tuple domination number. Let S be γ_{KT} set for graph G .

- (1) Let $u \notin S, v \notin S$. Then S is a K -tuple dominating set in $G-\{e\}$. Hence $\gamma_{KT}(G-e) \leq \gamma_{KT}(G)$. Which is a contradiction.
 (2) Let $u \in S, v \notin S$. (a) Suppose v is adjacent to at least $K+1$ vertices of S including u . Then removing $e=uv$ still v will be adjacent to at least K vertices of S and $e=uv \notin E_{KT}^+$. Which is a contradiction. (b) Suppose v is adjacent to at most $K-1$ vertices of S including u . Then S is not a K -tuple dominating set. Which is a contradiction.
 So, $e=uv$ then $u \in S, v \notin S$ and v is adjacent to exactly K vertices of S including u .

- (3) Let $u \in S, v \in S$ (a) Suppose v is adjacent to at least K vertices of S including u . Then removing $e=uv$ still v will be adjacent to at least $K-1$ vertices of S and $e=uv \notin E_{KT}^+$ Which is a contradiction. (b) Suppose v is adjacent to at most $K-2$ vertices of S including u . Then S is not a K -tuple dominating set Which is a contradiction. So, $e=uv$ then $u \in S, v \in S$ and v is adjacent to exactly $K-1$ vertices of S including u . Similarly u is adjacent to exactly $K-1$ vertices of S including v .

Theorem 7: An edge $e=uv \in E_{KT}^0 \Leftrightarrow$ There exist a γ_{KT} set S of graph G such that

- (1) $u \notin S, v \notin S$
- (2) $u \in S, v \notin S$ and v is adjacent to at least $K+1$ vertices of S .
- (3) $u \in S, v \in S$ then u and v are adjacent to at least K vertices of S .

Proof: $\Leftarrow$ (Sufficient) Suppose one of these conditions hold.

Then if we remove edge $e=uv$ then still S is a K -tuple dominating set in graph $G-\{e\}$ also.

Therefore S is a minimum K -tuple dominating set in $G-\{e\}$. so,

$$\gamma_{KT}(G-\{e\}) = |S| = \gamma_{KT}(G) \text{ . So, } e=uv \in E_{KT}^0 \text{ .}$$

$\Rightarrow$ (Necessity)

Let $e=uv \in E_{KT}^0$ means removal of $e=uv$ does not affect K -tuple domination number. Let S be a γ_{KT} set of $G-\{e\}$ then $|S| = \gamma_{KT}(G)$.

Let (1) $u \notin S, v \notin S$ then theorem is proved.

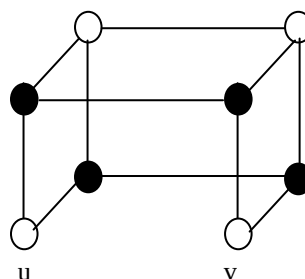
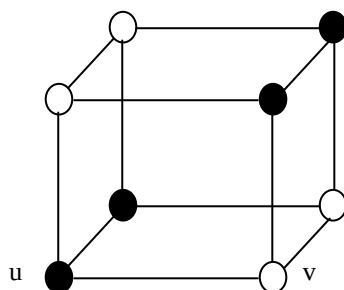
Let (2) $u \in S, v \notin S$. So, v is adjacent to at least K vertices of S in $G-\{e\}$. Hence v is adjacent to at least $K+1$ vertices including u of S in G . Thus S is a γ_{KT} set of G . Which satisfies required Conditions.

Let (3) $u \in S, v \in S$. Now u and v are adjacent to at least $K-1$ vertices of S in $G-\{e\}$. Hence u and v are adjacent to at least K vertices of S in G . Thus S is a γ_{KT} set of G which Satisfies required conditions.

Example 8: Here we give an example of a graph which represents K -tuple ($K=2$) dominating set and edge whose removal does not affect K -tuple ($K=2$) domination number.

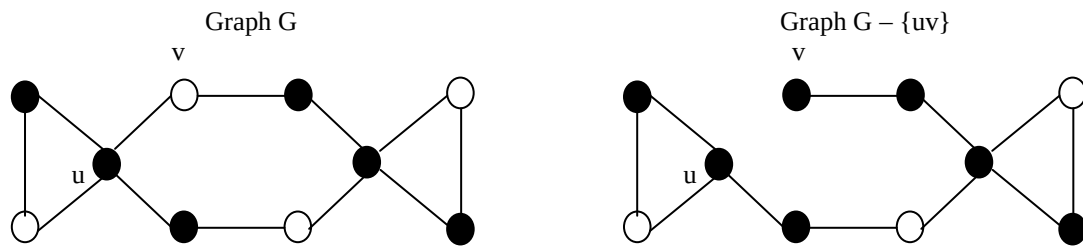
Graph G

Graph $G-\{uv\}$



here we can see that all edges $e=uv \in E_{KT}^0$

Example 9: Here we give an example of a graph which represents K -tuple ($K=2$) dominating set and edge whose removal Increases K -tuple ($K=2$) domination number.



here we can see that all edges $e=uv \in E_{KT}^+$

Note: Now we consider the case of adding an edge between two non-adjacent vertices of given graph. It may be noted that in this case K - tuple domination number of graph will not Increase .Moreover In this section we assume that the Minimum degree of given graph is at least K .

Theorem 10: Let u and v be two non – adjacent vertices of graph G and $e=uv$ then

$\gamma_{KT}(G+\{e\}) < \gamma_{KT}(G) \iff$ There is a subset S of $V(G)$ such that $|S| < \gamma_{KT}(G)$ and following conditions hold.

- (1) (a) $u \in S, v \notin S$ and v is adjacent to exactly $K-1$ vertices of S .
 (b) S is a K – tuple dominating set in the sub graph $G-\{v\}$.
- (2) (a) $u \in S, v \in S$ and v is adjacent to exactly $K-2$ vertices of S .
 (b) S is a K – tuple dominating set in the sub graph $G-\{v\}$.

Proof: $\Leftarrow$ (Sufficient)

Let S be subset of $V(G)$ such that $|S| < \gamma_{KT}(G)$ and one of given conditions hold.

S is a K – tuple dominating set in $G-\{v\}$ but not in G . So, $S+\{v\}$ will be K – tuple dominating set in graph G . If we add $e=uv$ then S will be K – tuple dominating set in $G+\{e\}$ and $|S| < \gamma_{KT}(G)$

So, $\gamma_{KT}(G+\{e\}) < \gamma_{KT}(G)$.

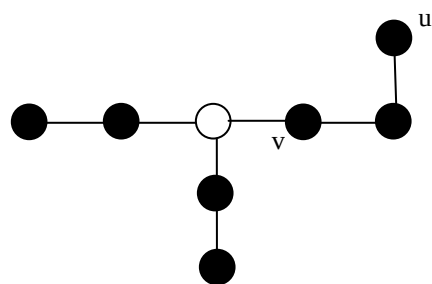
$\Rightarrow$ (Necessity)

Let S be a γ_{KT} set in $G+\{e\}$ then $|S| < \gamma_{KT}(G)$.

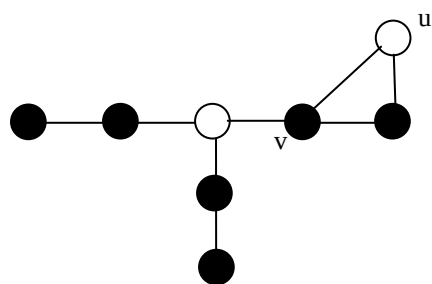
Let u and v be two non – adjacent vertices and $\gamma_{KT}(G+\{uv\}) < \gamma_{KT}(G)$.

- (1) Suppose $u \notin S, v \notin S$.Here S is a K – tuple dominating set in G also which is not possible as $|S| < \gamma_{KT}(G)$.
- (2) Suppose $u \in S, v \notin S$. Now by theorem 6 , v is adjacent to exactly K vertices of S including u in $G+\{e\}$. v is adjacent to exactly $K-1$ vertices of S .
- (3) Suppose $u \in S, v \in S$. Now by theorem 6 , v is adjacent to exactly $K-1$ vertices of S including u in $G+\{e\}$. v is adjacent to exactly $K-2$ vertices of S .

Example 11: Here we give an example of a graph which represents $K=2$ tuple dominating set and effect of edge addition between two non – adjacent vertices of graph. Here $\gamma_{2T}(G+uv) < \gamma_{2T}(G)$



Graph G



Graph G + {uv}

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