



Optimal control of an N-policy two-phase $M^X/E_k/1$ gated queueing system with server startup subject to the server breakdowns and delayed repair

B. Srinivasa Kumar, V. Vasanta Kumar, T.Srinivasa Rao
K.L. University, Vaddeswaram – 522502, Guntur (Dist), Andhra Pradesh, India.

Abstract: This paper deals with the economic behavior of an N-policy $M^X/E_k/1$ queue with server startup, two-phases of compulsory service with gating and unreliable server, which consists of a breakdown period and a delay period. Explicit expressions for the steady state distribution of the number of customers at different states of the customer in the system are obtained and the expected system length is derived. The mean queue waiting time by heuristic interpretation and the reliability indices of this model are also derived. This paper also dealt with the total expected cost function which is developed to determine the optimal threshold of N at a minimum cost. Sensitivity analysis is presented through numerical illustrations..

Keywords: two-phase, vacation, breakdowns, N-policy, repair time, delay time, cost function, gating.

I. Introduction

The service station in many queueing systems (eg: in communication systems, computer networks, flexible manufacturing systems etc) is subject to unpredictable breakdowns and can be repaired. Therefore, it is necessary to see how the breakdowns affect the level of performance of the system. Regarding the queueing systems with two phases of service, Krishna and Lee [4] and Doshi[3] studied the distributed systems where all customers waiting in the queue receive batch service in the first phase of service followed by individual service in the second phase. Vasanta Kumar and Chandan [5] presented the Optimal Operating policy for the two-phase $M^X/E_k/1$ queueing system under N-policy. Vasanta Kumar et al.[6,7] presented the optimal Control of $M^X/M/1$ and $M/E_k/1$ gated queueing systems with server startup and breakdowns. Vasanta Kumar et al.[8,9] presented the Optimal operating policy for a two-phase $M^X/M/1$ and $M^X/E_k/1$ queueing systems under N-policy with server breakdowns and without gating. Choudhury and Tadj [1] considered an M/G/1 model with an additional phase of optional service with the assumption that the server is subject to breakdowns and delayed repair. Later Choudhury et al. [2] investigated such a type of model, where the concept of N-policy is also introduced along with a delayed repair for batch arrival queueing system.

Thus the present study is aimed at the study of the economic behavior of an N-policy $M^X/E_k/1$ gated queue with service is in two phases and the server is typically subject to unpredictable breakdowns and delay in repair.

II. The Mathematical Model

(i) The arrival process is a compound Poisson process(with λ) of independent and identically distributed random batches of customers, where each batch size X, has a probability density function{ a_n ; $a_n = P(X=n)$, $n \geq 1$ }. Batches are admitted to service on a first come first served basis.

(ii) The service is in two phases. The first phase of service is batch service to all customers waiting in the queue. On completion of batch service the server proceeds to second phase to serve all customers in the batch individually. Batch service time is assumed to be exponentially distributed with mean $1/\beta$ and is independent of batch size. Individual service is in k independent and identically distributed exponential phases with mean $1/k\mu$. On completion of individual service, the server returns to the batch queue to serve the customers who have arrived. If customers are waiting, the server starts the batch service followed by individual service to each customer in the batch. If no customer is waiting, the server takes a vacation.

(iii) Whenever, the system becomes empty, the server is turned off. As soon as the total number of arrivals in the queue reaches to a predetermined threshold N the server is turned on and is temporarily unavailable for the waiting customers to restart service. Before the batch service, the server needs a startup time which follows an

exponential distribution with mean $1/\theta$. As soon as the server finishes startup, it starts serving the first phase of waiting customers.

(iv) The customers who arrive during the batch service are not allowed to join the same batch which is in service but are served in the next visit of the server to the batch queue.

(v) The server is subjected to unpredictable breakdowns during individual service with Poisson breakdown rate α . Whenever the server fails it is sent for repair during which the server stops providing the service and wait for the repair to get started. The waiting time for repair is defined as delay time and assumed to be exponentially distributed with mean $1/\delta$. Repair time is also assumed to be exponentially distributed with mean $1/\gamma$.

III. Steady State results.

In steady state the following notations are used.

$P_{0,i,0}$ = The probability that there are i customers in the batch queue when the server is on vacation, $i = 0, k, 2k, \dots, (N-1)k$

$P_{1,i,0}$ = The probability that there are i customers in the batch queue when the server is doing pre-service (startup work), where $i = Nk, (N+1)k, (N+2)k, \dots$

$P_{2,i,0}$ = The probability that there i customers in the batch which is in batch service, $i = k, 2k, 3k, \dots$

$P_{3,i,j}$ = The probability that there are i customers in the batch queue and j customers in the individual queue when the server is in individual service, $i = k, 2k, 3k, \dots$, and $j = 1, 2, 3, \dots$

$P_{4,i,j}$ = The probability that there are i customers in the batch queue and j customers in the individual queue when the server is in individual service, but found to be broken down and waiting for repair, $i = k, 2k, 3k, \dots$, and $j = 1, 2, 3, \dots$

$P_{5,i,j}$ = The probability that there are i customers in the batch queue and j customers in the individual queue when the server is in individual service, but is under repair, $i = k, 2k, 3k, \dots$, and $j = 1, 2, 3, \dots$

The steady state equations satisfied by the system size probabilities are as follows:

$$\lambda P_{0,0,0} = k \mu P_{3,0,1}, \quad (1)$$

$$\lambda P_{0,i,0} = \lambda \sum_{l=k}^i c_l P_{0,i-l,0}, \quad k \leq i \leq (N-1)k. \quad (2)$$

$$(\lambda + \theta) P_{1,Nk,0} = \lambda \sum_{l=k}^{Nk} c_l P_{0,Nk-l,0}. \quad (3)$$

$$(\lambda + \theta) P_{1,i,0} = \lambda \sum_{l=k}^{i-Nk} c_l P_{1,i-l,0} + \lambda \sum_{l=i-(N-1)k}^i c_l P_{0,i-l,0}, \quad i > Nk. \quad (4)$$

$$\beta P_{2,i,0} = k \mu P_{3,i,1}, \quad k \leq i \leq (N-1)k, \quad (5)$$

$$\beta P_{2,i,0} = k \mu P_{3,i,1} + \theta P_{1,i,0} \quad i \geq Nk, \quad (6)$$

$$(\lambda + \alpha + k \mu) P_{3,0,j} = k \mu P_{3,0,j+1} + \pi_0 \beta P_{2,j,0} + \gamma P_{5,0,j}, \quad j \geq 1. \quad (7)$$

$$(\lambda + \alpha + k \mu) P_{3,i,j} = k \mu P_{3,i,j+1} + \pi_{i/k} \beta P_{2,j,0} + \lambda \sum_{l=k}^i c_l P_{3,i-l,j} + \gamma P_{5,i,j}, \quad i \geq k, j \geq 1. \quad (8)$$

$$(\lambda + \delta) P_{4,0,j} = \alpha P_{3,0,j}, \quad j \geq 1. \quad (9)$$

$$(\lambda + \delta) P_{4,i,j} = \alpha P_{3,i,j} + \lambda \sum_{l=k}^i c_l P_{4,i-l,j}, \quad i \geq k, j \geq 1. \quad (10)$$

$$(\lambda + \gamma) P_{5,0,j} = \delta P_{4,0,j}, \quad j \geq 1. \quad (11)$$

$$(\lambda + \gamma) P_{5,i,j} = \delta P_{4,i,j} + \lambda \sum_{l=k}^i c_l P_{5,i-l,j}, \quad i \geq k, j \geq 1. \quad (12)$$

Where $\Pi_{i/k} = \frac{\left(\frac{\lambda C'(1)}{k}\right)^{i/k} \beta}{\left(\frac{\lambda C'(1)}{k} + \beta\right)^{(i/k)+1}}$ and c_l is the probability that there are l -service phases in an arrival batch.

$C(z) = \sum_{i=1}^{\infty} c_i z^i$ is the probability generating function of the number of service phases in the batch and $C'(z)$,

$C''(z)$ represent the first and second order derivatives of $C(z)$, respectively. The following probability generating functions are used to solve the equations (1) to (12).

$$G_0(z) = \sum_{i=0}^{(N-1)k} P_{0,i,0} z^i, \quad G_1(z) = \sum_{i=Nk}^{\infty} P_{1,i,0} z^i, \quad G_2(z) = \sum_{i=k}^{\infty} P_{2,i,0} z^i, \quad G_3(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{3,i,j} z^i y^j$$

$$G_4(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{4,i,j} z^i y^j, \quad G_5(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{5,i,j} z^i y^j, \quad R_j(z) = \sum_{i=0}^{\infty} P_{3,i,j} z^i, \quad S_j(z) = \sum_{i=0}^{\infty} P_{4,i,j} z^i$$

$$T_j(z) = \sum_{i=0}^{\infty} P_{5,i,j} z^i \quad \text{and} \quad \Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i \quad \text{where } |z| \leq 1 \text{ and } |y| \leq 1.$$

Solving equations (1) to (12) the following generating functions are obtained:

$$G_0(z) = P_{0,0,0} Y_N(z), \text{ where } Y_N(z) = \sum_{i=0}^{(N-1)k} y_i z^i, \quad y_i = \sum_{l=k}^i c_l P_{0,i-l,0} z^i \quad (13)$$

$$[\lambda(1 - C(z)) + \theta] G_1(z) = \lambda P_{0,0,0} + \lambda(C(z) - 1) G_0(z) \quad (14)$$

$$\beta G_2(z) = k \mu R_1(z) - k \mu P_{3,0,1} + \theta G_1(z). \quad (15)$$

$$[\lambda(1 - C(z))y + \alpha y - k \mu(1 - y)] G_3(z, y) = \beta y \Pi(z) G_2(y) + \gamma y G_5(z, y) - k \mu y R_1(z) \quad (16)$$

$$[\lambda(1 - C(z)) + \delta] G_4(z, y) = \alpha G_3(z, y). \quad (17)$$

$$[\lambda(1 - C(z)) + \gamma] G_5(z, y) = \delta G_4(z, y). \quad (18)$$

The total probability generating function $G(z, y)$ is given by

$$G(z, y) = G_0(z) + G_1(z) + G_2(z) + G_3(z, y) + G_4(z, y) + G_5(z, y). \quad (19)$$

The normalizing condition is

$$G(1, 1) = G_0(1) + G_1(1) + G_2(1) + G_3(1, 1) + G_4(1, 1) + G_5(1, 1) = 1. \quad (20)$$

This condition yields, $R_1(1) = \lambda C'(1) / k^2 \mu$

Under steady state conditions, let $P_v, P_s, P_b, P_i, P_{bi}$ and P_{di} be the probabilities that the server is in vacation, in startup, in batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively. Then

$$P_v = G_0(1) = Y_N(1) P_{0,0,0}, \quad (21)$$

$$P_s = G_1(1) = (\lambda P_{0,0,0} / \theta), \quad (22)$$

$$P_b = G_2(1) = (\lambda C'(1) / k \beta), \quad (23)$$

$$P_i = G_3(1, 1) = (\lambda C'(1) / k \mu), \quad (24)$$

$$P_{bi} = G_4(1, 1) = (\alpha \lambda C'(1)) / (k \gamma \mu), \quad (25)$$

$$P_{di} = G_5(1, 1) = (\alpha \lambda C'(1)) / (k \delta \mu), \quad (26)$$

where $\left(\frac{\lambda + Y_N(1)\theta}{\theta}\right) P_{0,0,0} = (1-\rho)$, and $\rho = \frac{\lambda C'(1)}{k\beta} + \frac{\lambda C'(1)}{k\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta}\right)$ is the utilizing factor of the system.

IV. Expected number of customers in different states

Using the probability generating functions expected numbers of customers in the system at different states are derived in this section. Let $L_v, L_s, L_b, L_i, L_{bi}$ and L_{di} be the expected number of customers in the system when the server is in vacation, in startup, in batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively. Differentiating the generating functions $G_q(z)$, $q = 0, 1, 2$ and $G_q(z, z)$, $q = 3, 4, 5$ given in equations (13) to (18) and substituting $z=1$ we get

$$L_v = G_0'(1) = Y_N(1) P_{0,0,0} \quad (27)$$

$$L_s = G_1'(1) = \frac{\lambda C'(1)(\lambda + \theta Y_N(1))}{\theta^2} P_{0,0,0} \quad (28)$$

$$L_b = G_2'(1) = \lambda C'(1) / \beta \quad (29)$$

$$\begin{aligned} L_i = G_3'(1,1) &= \frac{\lambda C'(1)}{k\mu} \left[\frac{1}{(1-\rho_1)} + \frac{\lambda C'(1)(\lambda + \theta Y_N(1))}{\theta^2(1-\rho_1)} P_{0,0,0} + \frac{Y_N(1)}{(1-\rho_1)} P_{0,0,0} \right. \\ &+ \left(\frac{C''(1)}{2C'(1)} \right) \left(\frac{\lambda + \theta Y_N(1)}{(1-\rho_1)\theta} \right) P_{0,0,0} - \frac{(\lambda C'(1))^2}{k\mu(1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) - \frac{(\lambda C'(1))^2 \alpha}{k\mu\gamma\delta(1-\rho_1)} \\ &+ \frac{\lambda C'(1)\rho_1}{(1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{\lambda C'(1)}{\beta(1-\rho_1)} + \frac{(\lambda C'(1))^2}{k\beta^2(1-\rho_1)} + \frac{C''(1)\lambda}{2k\mu(1-\rho_1)} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \\ &\left. - \lambda C'(1) \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{(k-1)\lambda C'(1)}{2\beta(1-\rho_1)} \right] \quad (30) \end{aligned}$$

$$L_{bi} = G_4'(1,1) = \frac{(\lambda C'(1))^2 \alpha}{(k^2 \delta^2 \mu)} + \frac{\alpha}{\delta} G_3'(1,1) \quad (31)$$

$$L_{di} = G_5'(1,1) = \frac{(\lambda C'(1))^2 \alpha}{k^2 \gamma^2 \mu} + \frac{(\lambda C'(1))^2 \alpha}{k^2 \mu \gamma \delta} + \frac{\alpha}{\gamma} G_3'(1,1), \quad (32)$$

where $\rho_1 = \frac{\lambda C'(1)}{k\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right)$ and $G_q'(\cdot)$ denote the first order derivative of $G_q(\cdot)$. Finally, the expected number of service phases in the system is given by

$$\begin{aligned} L(P) &= G_0'(1) + G_1'(1) + G_2'(1) + G_3'(1,1) + G_4'(1,1) + G_5'(1,1) \\ &= \frac{\rho_1}{(1-\rho_1)} + \frac{Y_N(1)}{(1-\rho_1)} P_{0,0,0} + \frac{\lambda C'(1)(\lambda + \theta Y_N(1))}{\theta^2(1-\rho_1)} P_{0,0,0} + \frac{\lambda C'(1)}{\beta(1-\rho_1)} \\ &+ \frac{(\lambda C'(1))^2 \rho_1}{k\beta^2(1-\rho_1)} - \frac{(\lambda C'(1))^2 \rho_1}{k\mu(1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{(\lambda C'(1))^2 \rho_1^2}{(1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) \\ &- \frac{(\lambda C'(1))^2 \alpha}{k\mu\gamma\delta} \left(\frac{\rho_1}{(1-\rho_1)} \right) + \frac{\rho_1 C''(1)}{2C'(1)(1-\rho_1)} - \frac{\lambda C''(1)\rho_1}{2k\beta(1-\rho_1)} \\ &+ \frac{(\lambda C'(1))^2 \alpha}{k^2 \mu \gamma \delta} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) + \frac{(k-1)\lambda C'(1)\rho_1}{2k\beta(1-\rho_1)} \quad (33) \end{aligned}$$

Let $A(z) = \sum_{i=0}^{\infty} a_i z^i$ be the probability generating function of the number of customers in the system and $A^1(z)$, $A^{11}(z)$ be the first and second derivatives of $A(z)$ respectively. It can be shown that $C^1(1) = k A^1(1)$ and $C^{11}(1) = K^2 A^{11}(1) + k(k-1) A^1(1)$. Expressing $L(P)$ in terms of $A^1(1)$ and $A^{11}(1)$

$$L(P) = \frac{\rho_1}{(1-\rho_1)} + \frac{Y_N'(1)}{(1-\rho_1)} P_{0,0,0} + \frac{\lambda k A^1(1) (\lambda + \theta Y_N(1))}{\theta^2 (1-\rho_1)} P_{0,0,0} + \frac{\lambda k A^1(1)}{\beta (1-\rho_1)} + \frac{k (\lambda A^1(1))^2 \rho_1}{\beta^2 (1-\rho_1)} - \frac{k (\lambda A^1(1))^2 \rho_1}{\mu (1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{k \lambda A^1(1) \rho_1^2}{(1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{(\lambda A^1(1))^2 \alpha}{\mu \gamma \delta} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) - \frac{k (\lambda A^1(1))^2 \alpha}{\mu \gamma \delta} \left(\frac{\rho_1}{(1-\rho_1)} \right) + \frac{k A''(1) + (k-1) A^1(1)}{2 A^1(1) (1-\rho_1)} - \frac{\lambda (k A''(1) + (k-1) A^1(1)) \rho_1}{2 \beta (1-\rho_1)} + \frac{(k-1) \lambda A^1(1) \rho_1}{2 \beta (1-\rho_1)}$$

(34)

Then the expected number of customers in the system is given by

$$L(N) = \frac{1}{k} \left[L(P) - \left(\frac{k+1}{2} \right) \left(\frac{\lambda A^1(1)}{\beta} + \frac{\lambda A^1(1)}{\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \right) \right] + \left[\frac{\lambda A^1(1)}{\beta} + \frac{\lambda A^1(1)}{\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \right]$$

(35)

where $\rho_1 = \frac{\lambda A^1(1)}{\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right)$ and $\rho = \frac{\lambda A^1(1)}{\beta} + \rho_1$ is the utilizing factor of the system.

V. Some Other system performance measures

In this section, expected length of vacation period, startup period, batch service period, individual service period, delay and repair periods during individual service are presented. The expected length of a busy cycle E_c is given by $E_c = E_v + E_s + E_b + E_i + E_{bi} + E_{di}$.

The long run fractions of time the server is in different states are respectively,

$$E_v / E_c = P_v = Y_N(1) P_{0,0,0}, E_s / E_c = P_s = (\lambda P_{0,0,0}) / \theta, E_b / E_c = P_b = \lambda A^1(1) / \beta, E_i / E_c = P_i = \lambda A^1(1) / \mu, E_{bi} / E_c = P_{bi} = (\alpha \lambda A^1(1)) / (\delta \mu) \text{ and } E_{di} / E_c = P_{di} = (\alpha \lambda A^1(1)) / (\gamma \mu).$$

Expected length of vacation period is given by $E_v = Y_N(1) / \lambda A^1(1)$,

Hence, $E_c = 1 / (\lambda A^1(1) P_{0,0,0})$.

VI. Specific batch size distribution.

Let X has geometric distribution. Then $a_j = P(X=j) = p(1-p)^{j-1}$, $0 < p < 1$, $j=1,2,3,\dots$ with the probability density function $A(z) = p z / [1 - (1-p)z]$ and $E(X) = A^1(1) = 1/p$, $E[X(X-1)] = A^{11}(1) = 2(1-p)/p^2$.

The expected number of customers in the system is given by

$$L(P) = \frac{\rho_1}{(1-\rho_1)} + \frac{Y_N'(1)}{(1-\rho_1)} P_{0,0,0} + \frac{\lambda k (\lambda + \theta Y_N(1))}{p \theta^2 (1-\rho_1)} P_{0,0,0} + \frac{\lambda k}{\beta p (1-\rho_1)} + \frac{k \lambda^2 \rho_1}{\beta^2 p^2 (1-\rho_1)} - \frac{k \lambda^2 \rho_1}{\mu p^2 (1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{k \lambda \rho_1^2}{p (1-\rho_1)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right)$$

$$+ \frac{\lambda^2 \alpha}{p^2 \mu \gamma \delta} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) - \frac{k \lambda^2 \alpha}{p^2 \mu \gamma \delta} \left(\frac{\rho_1}{1 - \rho_1} \right) + \frac{2k(1-p) + (k-1)p}{2p(1-\rho_1)} - \frac{\lambda(2(1-p)k + (k-1)p)\rho_1}{2\beta p^2(1-\rho_1)} + \frac{\lambda(k-1)}{2p\beta} \left(\frac{\rho_1}{1-\rho_1} \right). \quad (36)$$

$$L(N) = \frac{1}{k} \left[L(P) - \left(\frac{k+1}{2} \right) \left(\frac{\lambda}{p\beta} + \frac{\lambda}{p\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \right) \right] + \left[\frac{\lambda}{p\beta} + \frac{\lambda}{p\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \right], \text{ where } \rho_1 = \frac{\lambda}{p\mu} \left(1 + \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right). \quad (37)$$

VII. Heuristic Approach to waiting time in the queue.

Let W_q be the waiting time of the test customer until his individual service. Any arbitrary customer waits different time amounts according to the state of his arriving epoch. First, we divide the regeneration cycle into six parts of the idle period, the startup period, the first phase batch service, the second phase individual service, the waiting time for repair due to breakdown in second phase and the repair time with respective probabilities

$$Y_N(1)P_{0,0,0}, \frac{\lambda P_{0,0,0}}{\theta}, \frac{\lambda A'(1)}{\beta}, \frac{\lambda A'(1)}{\mu}, \frac{\lambda A'(1)}{\mu} \left(\frac{\alpha}{\delta} \right), \text{ and } \frac{\lambda A'(1)}{\mu} \left(\frac{\alpha}{\gamma} \right).$$

That is the system state that the arriving customer sees determines his waiting time. The test customer has to wait during the individual service times for those already waiting (except the ongoing individual service) in the system. In addition to it,

- (i) If the server is idle, the customer has to wait the remaining idle period, startup period and the first phase batch service period.
- (ii) If the server is in startup state, the customer has to wait the remaining startup period and the first phase batch service period.
- (iii) If the server is in the first phase, the customer has to wait the remaining time of the ongoing batch service and its individual service period plus the batch service period in which he gets batch service.
- (iv) If the server is in the second phase of service, the customer has to wait the remaining time of the ongoing individual service plus the batch service period.
- (v) If the server is waiting for repair due to breakdown during individual service, the customer has to wait the remaining waiting period, the repair period and the first phase batch service period.
- (vi) If the server is in repair state due to breakdown, the customer has to wait the remaining repair period and the first phase batch service period.

Thus

$$E(w_q) = \frac{1}{1-r} \left[\left(\frac{N-1}{2\lambda A'(1)} + \frac{1}{\theta} + \frac{1}{\beta} \right) Y_N(1)P_{0,0,0} + \left(\frac{1}{\theta} + \frac{1}{\beta} \right) \left(\frac{\lambda}{\theta} P_{0,0,0} \right) + \left(\frac{2\lambda A'(1)}{\beta^2} \right) + \left(\frac{1}{\mu} + \frac{1}{\beta} \right) \frac{\lambda A'(1)}{\mu} + \left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{1}{\beta} \right) \frac{\lambda \alpha A'(1)}{\mu \gamma} + \left(\frac{1}{\gamma} + \frac{1}{\beta} \right) \frac{\lambda A'(1)}{\mu \gamma} \right]. \quad (38)$$

$$\text{where, } r = \frac{\lambda A'(1)}{\mu} + \frac{\lambda A'(1)}{\beta}$$

VIII. Reliability indices.

IX.

In this section two reliability indices of the system viz.- the system availability and failure frequency under the steady state conditions are discussed.

The system availability for a customer is the probability that the server is working for a customer or in idle period, or in startup period, or in batch service. The steady state availability of the server will be given by

$$G_0(1) + G_1(1) + G_2(1) + G_3(1,1) = 1 - \frac{\lambda A'(1)}{\mu} \left(\frac{\alpha}{\gamma} + \frac{\alpha}{\delta} \right) \quad (39)$$

The steady state failure frequency of the server is given by

$$F = \alpha G_3(1,1) = \frac{\lambda \alpha A'(1)}{\mu} \quad (40)$$

X. Optimal Cost structure

In this section the optimal value of N that minimizes the long run average cost of two- phase $M^X/E_k/1$, N -policy gated queue with server break downs and delay in repair is determined.

To determine the optimal value of N we consider the following linear cost structure is considered. Let $T_A(N)$ be the average cost per unit of time, then

$$T_A(N) = C_h L(N) + C_0 \left(\frac{E_b}{E_c} + \frac{E_i}{E_c} \right) + C_m \left(\frac{E_s}{E_c} \right) + C_b \left(\frac{E_{bi} + E_{di}}{E_c} \right) + C_s \left(\frac{1}{E_c} \right) - C_r \left(\frac{E_v}{E_c} \right) \quad (41)$$

where

$C_h \equiv$ Holding cost per unit time for each customer present in the system,

$C_0 \equiv$ Cost per unit time for keeping the server on and in operation,

$C_m \equiv$ Start up cost per unit time per cycle,

$C_s \equiv$ Setup cost per cycle,

$C_b \equiv$ Break down cost per unit of time for the unavailable server, and

$C_r \equiv$ Reward per unit of time as the server is doing secondary work in vacation.

It is observed that $E_b / E_c, E_i / E_c, E_{bi} / E_c, E_{di} / E_c$ are not functions of the decision variable N . Hence for determination of the optimal operating N -policy, minimizing $T_A(N)$ in equation (41) is equivalent to minimizing.

$$T_A(N) = \frac{(1-\rho) \left[C_h Y_N'(1) + (1-\rho_1) \left(C_m \frac{\lambda}{\theta} + \lambda C_s - C_r Y_N(1) \right) \right]}{(1-\rho_1) \left(Y_N(1) + \frac{\lambda}{\theta} \right)}$$

It is hard to prove that $T_A(N)$ is convex. But a procedure that makes it possible to calculate the optimal threshold N^* is presented below.

Result:

Under the long run expected average cost criterion, the optimal threshold N^* for the model is the best value of ' k ' given by

$$N^* = \min \left\{ k \geq \sum_{j=0}^N (k-j) y_j + \frac{k\lambda}{\theta} > \frac{\lambda(1-\rho_1)}{C_h} \left(\frac{C_m - C_r}{\theta} + C_s \right) \right\}$$

and it is one of the integers surrounding N . see the Ref. (9) for proof.

XI. Sensitivity Analysis

We carry out numerical computations for various combinations of the system parameters, setting the following cost elements $C_h = 5, C_0 = 50, C_m = 100, C_b = 100, C_r = 40$ and $C_s = 1000$.

Table1. The optimal management policy and the minimum expected cost for $(\beta, \theta, \gamma, \alpha, \delta, k) = (4, 3, 3, 0, 2, 2)$

(λ, μ)	N^*	$T(N^*)$	(λ, μ)	N^*	$T(N^*)$
(0.2,3.5)	15	30.56	(0.4,2.0)	14	68.31
(0.3,3.5)	17	46.51	(0.4,2.5)	16	64.37
(0.4,3.5)	18	59.53	(0.4,3.0)	17	61.58
(0.5,3.5)	17	70.06	(0.4,3.5)	18	59.53
(0.6,3.5)	16	78.07	(0.4,4.0)	18	57.96
(0.7,3.5)	13	83.00	(0.4,4.5)	19	56.72

Table 2. The optimal management policy and the minimum expected cost for
 $(\beta, \theta, \gamma, \alpha, \delta, k) = (0.4, 3.5, 3, 0.2, 2, 2)$

(β, θ)	N^*	$T(N^*)$	(β, θ)	N^*	$T(N^*)$
(3.0,3.0)	16	62.85	(4.0,2.0)	18	60.23
(4.0,3.0)	18	59.53	(4.0,3.0)	18	59.53
(5.0, 3.0)	18	57.47	(4.0,4.0)	18	59.98
(6.0, 3.0)	19	56.05	(4.0,5.0)	18	59.53
(7.0, 3.0)	19	55.03	(4.0,6.0)	18	58.86s

From Table 1 it may be observed that (a) N^* is convex with increase in λ and shows increasing trend with increase in μ , and (b) $T(N^*)$ increases with increase in λ and decreases with increase in μ .

It can be seen from Table 2 that (a) N^* shows increasing trend with increase in β and insensitive with increase in θ , and (b) $T(N^*)$ decreases with increase in both β and θ .

We also performed same sensitivity analysis on the optimal value N^* based on changes in considerable values of other parameters γ, α, δ and the cost parameters C_h, C_o, C_m, C_b, C_s and C_r . It is noted that the results are almost identical to previous investigations.

XII. Conclusions

In this paper some important performance measures are derived for the N-policy $M^X/E_k/1$ gated queueing system with two phases of service, startup, breakdowns and delayed repair. As the convexity of the expected cost function cannot be proved theoretically, a heuristic approach is chosen to determine the optimal threshold. The mean waiting time and the reliability indices of this model are also derived. Sensitivity analysis is performed between the optimal value of N , system parameters and the cost elements when the arrival batch size distribution is geometric.

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