



Normal Families and Exceptional Functions

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Abstract: In this paper some normality criteria for a family of meromorphic functions are proved and these also generalize the result of Lie et al. [1] and Clifford [2].

Keywords: Meromorphic functions; Normal family; Differential polynomial; Transcendental entire functions; Value distribution.

I. Introduction

Let D be a domain in the whole complex plane. A family $\mathcal{F}$ is said to be normal in D , in the sense of Montel [3, 4, 5], if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D to a meromorphic function or to ∞ .

In 1979 Gu [6] proved the following normality criterion.

Theorem A. Let $\mathcal{F}$ be a family of meromorphic functions and b be a nonzero complex value. Then for any positive integer n and for each $f \in \mathcal{F}$, if $f \neq 0$ and $f^{(k)} \neq b$ then $\mathcal{F}$ is normal in D .

In 1998 Wang and Fang [7] improved the above theorem as follows

Theorem B. Let k be a positive integer, b be a nonzero complex value and let $\mathcal{F}$ be a family of meromorphic functions in a domain D , all of whose zeros have multiplicity at least $k + 2$. If for each function $f \in \mathcal{F}$, $f^{(k)}(z) \neq b$ then $\mathcal{F}$ is normal in D .

In the same paper Wang and Fang [7] further improved Theorem A by considering zeros and poles simultaneously as follows

Theorem C. Let k be a positive integer, b be a nonzero complex value and let $\mathcal{F}$ be a family of meromorphic functions in a domain D , all of whose poles are multiple and all of whose zeros have multiplicity at least $k + 1$. If for each function $f \in \mathcal{F}$, $f^{(k)}(z) \neq b$ then $\mathcal{F}$ is normal in D .

In 2002, Pang et al.[8], Xu [9] improved Theorem B by replacing the nonzero complex value with a holomorphic function having multiple zeros as follows.

Theorem D. Let k be a positive integer, $h(\not\equiv 0)$ be a holomorphic function which has only multiple zeros in D and let $\mathcal{F}$ be a family of meromorphic functions in a domain D , all of whose zeros have multiplicity at least $k + 2$. If for each function $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$ then $\mathcal{F}$ is normal in D .

In 2008 Lie et. al [1] enhanced Theorem C by introducing a holomorphic function h instead of a constant as follows

Theorem E. Let k be a positive integer with $k \geq 2$ and $h(\not\equiv 0)$ be a holomorphic function which has no simple zeros in a domain D and let $\mathcal{F}$ be a family of meromorphic functions defined in D , all of whose poles are multiple and all of whose zeros have multiplicity at least $k + 1$. If for each function $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$ then $\mathcal{F}$ is normal in D .

In the present paper the Theorem E is generalized by considering a differential linear polynomial as follows

Theorem 1. Let k be a positive integer with $k \geq 2$ and $h(\not\equiv 0)$ be a holomorphic function which has no simple zeros in a domain D and let $\mathcal{F}$ be a family of meromorphic functions defined in D , all of whose poles are multiple and all of whose zeros have multiplicity at least $k + 1$. If for each function $f \in \mathcal{F}$,

$$f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + \dots + a_0(z)f(z) \neq h(z),$$

where $a_i(z)$ are analytic in D for $i = 0, 1, \dots, k - 1$, then $\mathcal{F}$ is normal in D .

Hincliffe [10] in 2003 proved a normality criterion in connection with composite functions. He proved the following theorem.

Theorem F. Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$ and let Ω be a domain in $\mathbb{C}$. If $\mathbb{C}^* \setminus f(\mathbb{C}) = 0, \{\infty\}$ or $\{\alpha, \beta\}$ where α, β are two distinct values in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ then the family $\mathcal{G} = \{g: g \text{ is analytic in } \Omega, f \circ g \text{ has no fixed point in } \Omega\}$

is a normal family in Ω .

In 2005 Clifford [2] proved the following theorem.

Theorem G. Let f be a transcendental entire function with $\rho(f) < \frac{1}{k}$, where $k \in \mathbb{N}$. Also let $a_0(z), a_1(z), \dots, a_{k-1}(z)$ and $a(z)$ be analytic function in a domain Ω . Then $\mathcal{G} = \{g: g \text{ is analytic in } \Omega \text{ and } (f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z) \text{ in } \Omega\}$ is a normal family in Ω .

In this paper the following theorem is proved.

Theorem 2. Let f be a transcendental entire function with $\rho(f) < \frac{1}{k}$, where $k \in \mathbb{N}$. Also let $a_0(z), a_1(z), \dots, a_{k-1}(z)$ and $a(z), b(z)$ be analytic functions in a domain Ω . Then

$\mathcal{G} = \{g: g \text{ is analytic in } \Omega \text{ and } (f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) - a(z)(f \circ g)^n(z) \neq b(z) \text{ in } \Omega\}$ is a normal family in Ω .

II. Some Important Lemmas

In this section some important lemmas are stated which are necessary in proving the theorems.

Lemma 1 [11, 12]. Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ all of whose zeros have multiplicity at least k and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0 , for each α ($0 \leq \alpha \leq k$), there exist points $z_n \in \Delta$, $z_n \rightarrow z_0$ and $f_n \in \mathcal{F}$ and positive numbers $\rho_n \rightarrow 0$ such that

$$\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \rightarrow g(\zeta)$$

locally uniformly with respect to spherical metric where g is a non constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k such that $g^\#(\zeta) \leq g^\#(0) = kA + 1$. In particular g has order at most 2.

Lemma 2 [7]. Let k be a positive integer and $f(z)$ be a meromorphic function of finite order on $\mathbb{C}$, all of whose poles are multiple and all of whose zeros have multiplicity at least $k + 1$. If $f^{(k)}(z) \neq 1$ then f is constant.

Lemma 3 [8, 13]. Let $g(z)$ be a transcendental meromorphic function of finite order on $\mathbb{C}$ and $P(z)$ be a polynomial such that $P(z) \not\equiv 0$. Suppose that all zeros of $g(z)$ have multiplicity at least $k + 1$. Then $g^{(k)}(z) - P(z)$ has infinitely many zeros.

Lemma 4 [1]. Let $m(\geq 2)$, $k(\geq 2)$ be two positive integers and let $Q(z)$ be a rational function all of whose zeros have multiplicity at least $k + 1$ and all of whose poles are multiple with the possible exception of $z = 0$. Then $Q^{(k)}(z) = z^m$ has a solution in $\mathbb{C}$.

Lemma 5 [2]. Let $k \in \mathbb{N}$ and Ω be the open disc $B(0,1)$. Let $a(z)$ be an analytic function on Ω and $\{f_n\}$ be a sequence of analytic functions on Ω such that $f_n(z) \neq a(z)$ on Ω and let z_n be a sequence of points tending to $z_0 \in \Omega$ and let $\{\rho_n\}$ be sequence of positive numbers tending to 0. Suppose g is an entire function such that

$$\lim_{n \rightarrow \infty} \rho_n^k f_n(z_n + \rho_n z) = g(z)$$

locally uniformly on $\mathbb{C}$. Then either $g \equiv 0$ on $\mathbb{C}$ or $g(z) \neq 0$ on $\mathbb{C}$.

Lemma 6 [12]. A family $\mathcal{G}$ of analytic functions in the open disc $B(0,1)$ is not normal at the origin if and only if there exists a sequence $g_n \in \mathcal{G}$, a sequence of points $z_n \rightarrow 0$, a sequence of positive numbers $\{\rho_n\}$ tending to 0 and a non constant entire function g on $\mathbb{C}$ such that

$$\lim_{n \rightarrow \infty} g_n(z_n + \rho_n z) = g(z)$$

spherically locally uniformly on $\mathbb{C}$ such that the spherical derivative of g is bounded and $g^\#(z) \leq g^\#(0) = 1$.

Lemma 7 [2]. Let f be a meromorphic function with bounded spherical derivative. Then order of f is at most 2.

Lemma 8 [2]. Let f be a transcendental entire function with $\rho(f) < 1$ and g is a non constant entire function, then $(f \circ g)'$ has infinitely many zeros.

Lemma 9 [2]. Let $k \in \mathbb{N}$ and f be a transcendental entire function such that $\rho(f) < \frac{1}{k}$. Suppose g is an entire function of finite order such that $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$. Then g is constant.

Lemma 10 [2]. If f is a transcendental entire function with $\rho(f) < 1$ then f' has infinitely many zeros.

III. Proof of Theorems

Proof of Theorem 1. The theorem is proved by considering two cases.

Case I. We first show that $\mathcal{F}$ is normal at a point z_0 with $h(z_0) \neq 0$. If possible suppose that $\mathcal{F}$ is not normal at z_0 . We may assume that $D = \Delta$ and $h(z_0) = 1$. Then by Lemma 1 there exists functions $f_n \in \mathcal{F}$ and sequence of points $z_n \rightarrow z_0$, sequence of positive numbers $\rho_n \rightarrow 0$ such that

$$\rho_n^{-k} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \rightarrow g(\zeta)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$ where $g(\zeta)$ is a non constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k + 1$ and all of whose poles are multiple. Again g is of order at most 2.

Noting that $a_i(z_n + \rho_n \zeta)g_n^{(i)}(\zeta)$ is locally bounded on $\mathbb{C}$ excepting the set of poles of $g(\zeta)$. Since

$$a_i(z_n + \rho_n \zeta) g_n^{(i)}(\zeta) \rightarrow a_i(z_0) g^{(i)}(\zeta)$$

on every compact subsets of $\mathbb{C}$ which contains no poles of $g(\zeta)$. So we have

$$\begin{aligned} g^{(k)}(\zeta) - 1 &= \lim_{n \rightarrow \infty} \left\{ g_n^{(k)}(\zeta) + \sum_{i=0}^{k-1} a_i(z_n + \rho_n \zeta) \rho_n^{k-i} g_n^{(i)}(\zeta) \right\} - h(z_0) \\ &= \lim_{n \rightarrow \infty} \left\{ f_n^{(k)}(z_n + \rho_n \zeta) + \sum_{i=0}^{k-1} a_i(z_n + \rho_n \zeta) f_n^{(i)}(z_n + \rho_n \zeta) - h(z_n + \rho_n \zeta) \right\}. \end{aligned}$$

Expression in the bracket does not vanish so by Hurwitz's Theorem either $g^{(k)}(\zeta) \equiv 1$ or $g^{(k)}(\zeta) \neq 1$. If $g^{(k)}(\zeta) \equiv 1$ then g must be a polynomial of exact degree k which contradicts the fact that each zero of g has multiplicity at least $k+1$. Hence $g^{(k)}(\zeta) \neq 1$. Then by Lemma 2, $g(\zeta)$ is constant. Again a contradiction. Hence $\mathcal{F}$ is normal at z_0 with $h(z_0) \neq 0$.

Case II. We now prove that $\mathcal{F}$ is normal at a point z_0 with $h(z_0) = 0$. Without loss of generality we may assume that $z_0 = 0$. Making the standard normalization we have

$$h(z) = z^m + b_{m+1} z^{m+1} + \dots = z^m b(z), \quad z \in \Delta \text{ where } m \geq 2, \quad b(0) = 1 \text{ and } h(z) \neq 0 \text{ for } 0 < |z| < 1.$$

Let $\mathcal{H} = \{F: F(z) = \frac{f(z)}{z^m}, f \in \mathcal{F}\}$. Since $f^{(k)}(0) + \sum_{i=0}^{k-1} a_i(0) f^{(i)}(0) \neq h(0) = 0$ and all zeros of $f(z)$ have multiplicity at least $k+1$, so $f(0) \neq 0$. Hence $F(0) = \infty$.

We shall prove that $\mathcal{H}$ is normal at 0. If possible, let $\mathcal{H}$ is not normal at 0. Then by Lemma 1, there exists $F_n \in \mathcal{H}$ and $z_n \rightarrow 0, \rho_n \rightarrow 0$ such that

$$\rho_n^{-k} F_n(z_n + \rho_n \zeta) = g_n(\zeta) \rightarrow g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$ where $g(\zeta)$ is a non constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$ and all of whose poles are multiple.

Let $f_n(z) = z^m F_n(z)$. We consider two sub cases.

Subcase I. We may suppose that $\frac{z_n}{\rho_n} \rightarrow \infty$. We have

$$\begin{aligned} f_n^{(k)}(z) &= z^m F_n^{(k)}(z) + \sum_{j=1}^k \binom{k}{j} (z^m)^{(j)} F_n^{(k-j)}(z) \\ &= z^m F_n^{(k)}(z) + \sum_{j=1}^k c_j (z^{m-j}) F_n^{(k-j)}(z) \end{aligned}$$

$$\text{where } c_j = \begin{cases} \binom{k}{j} m(m-1)\dots(m-j-1), & j \leq m \\ 0, & j > m \end{cases}.$$

Now $f_n^{(k)}(z) = z^m \left\{ F_n^{(k)}(z) + \sum_{j=1}^k c_j \frac{F_n^{(k-j)}(z)}{z^j} \right\}$ and since $\rho_n^{k-i} g_n^{(i)}(\zeta) = F_n^{(i)}(z_n + \rho_n \zeta)$ for $i = 0, 1, 2, \dots, k$, so

$$\begin{aligned} f_n^{(i)}(z_n + \rho_n \zeta) &= (z_n + \rho_n \zeta)^m \left\{ \rho_n^{(k-i)} g_n^{(i)}(\zeta) + \sum_{j=1}^i c_j \frac{\rho_n^{k-i+j} g_n^{(i-j)}(\zeta)}{(z_n + \rho_n \zeta)^j} \right\} \\ &= (z_n + \rho_n \zeta)^m \rho_n^{(k-i)} \left\{ g_n^{(i)}(\zeta) + \sum_{j=1}^i c_j \frac{g_n^{(i-j)}(\zeta)}{\left(\frac{z_n + \rho_n \zeta}{\rho_n}\right)^j} \right\} \text{ for } i = 0, 1, 2, \dots, k-1. \end{aligned}$$

Again $\lim_{n \rightarrow \infty} \frac{c_j}{\left(\frac{z_n + \rho_n \zeta}{\rho_n}\right)^j} = 0, j = 1, 2, \dots, k$ and as $\lim_{n \rightarrow \infty} \frac{1}{b(z_n + \rho_n \zeta)} = \frac{1}{b(0)} = 1$, so

$$\frac{f_n^{(i)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} = \frac{1}{b(z_n + \rho_n \zeta)} \rho_n^{(k-i)} \left\{ g_n^{(i)}(\zeta) + \sum_{j=1}^i c_j \frac{g_n^{(i-j)}(\zeta)}{\left(\frac{z_n + \rho_n \zeta}{\rho_n}\right)^j} \right\}$$

and hence $\frac{f_n^{(i)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \rightarrow 0$ for $i = 0, 1, 2, \dots, k-1$.

Therefore

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta) + \sum_{i=0}^{k-1} a_i(z_n + \rho_n \zeta) f_n^{(i)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \rightarrow g^{(k)}(\zeta)$$

converges spherically uniformly on compact subset of $\mathbb{C}$ excepting the set of poles of g where $a_i(z_n + \rho_n \zeta)$ are analytic in D . Since

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta) + \sum_{i=0}^{k-1} a_i(z_n + \rho_n \zeta) f_n^{(i)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \neq 1$$

then by Hurwitz's theorem, either $g^{(k)}(\zeta) \equiv 1$ or $g^{(k)}(\zeta) \neq 1 \quad \forall \zeta \in \mathbb{C}$. If $g^{(k)}(\zeta) \equiv 1$ then g must be a polynomial of exact degree k which contradicts the fact that each zero of g has multiplicity at least $k+1$. Hence $g^{(k)}(\zeta) \neq 1$. Then by Lemma 2, $g(\zeta)$ is constant. Again a contradiction. Hence $\mathcal{H}$ is normal at z_0 with $h(z_0) \neq 0$.

Subcase II. When $\frac{z_n}{\rho_n} \rightarrow \alpha$, a finite number.

We have

$$\frac{F_n(\rho_n \zeta)}{\rho_n^k} = \frac{F_n(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))}{\rho_n^k} \rightarrow g(\zeta - \alpha)$$

converges spherically uniformly on compact subset of $\mathbb{C}$. Clearly all zeros of $g(\zeta - \alpha)$ have multiplicity at least $k + 1$ and poles of $g(\zeta - \alpha)$ at $\zeta = 0$ is of multiplicity at least m . Now

$$G_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^{m+k}} = \frac{F_n(\rho_n \zeta)}{\rho_n^k} \frac{(\rho_n \zeta)^m}{\rho_n^m} \rightarrow \zeta^m g(\zeta - \alpha) = G(\zeta)$$

converges uniformly on compact subsets of $\mathbb{C}$. Since $g(\zeta - \alpha)$ has a pole of multiplicity at least m at $\zeta = 0$ and $G(0) \neq 0$ and all zeros of $G(\zeta)$ have multiplicity at least $k + 1$. We claim that $G^{(k)}(\zeta) \neq \zeta^m$. If possible, let there exists ζ_0 such that $G^{(k)}(\zeta_0) = \zeta_0^m$. Then $G(\zeta)$ is holomorphic at ζ_0 and

$$G_n^{(k)}(\zeta) + \sum_{i=0}^{k-1} a_i(\rho_n \zeta) \rho_n^{k-i} G_n^{(i)}(\zeta) - \frac{h((\rho_n \zeta))}{\rho_n^m} = \frac{f_n^{(k)}(\rho_n \zeta) + \sum_{i=0}^{k-1} a_i(\rho_n \zeta) f_n^{(i)}(\rho_n \zeta) - h(\rho_n \zeta)}{\rho_n^m} \neq 0.$$

Now

$$\begin{aligned} G_n^{(k)}(\zeta) + \sum_{i=0}^{k-1} a_i(\rho_n \zeta) \rho_n^{k-i} G_n^{(i)}(\zeta) - \frac{h((\rho_n \zeta))}{(\rho_n \zeta)^m} \zeta^m &\neq 0 \\ \Rightarrow G_n^{(k)}(\zeta) + \sum_{i=0}^{k-1} a_i(\rho_n \zeta) \rho_n^{k-i} G_n^{(i)}(\zeta) - \frac{\zeta^m}{b(\rho_n \zeta)} &\neq 0 \end{aligned}$$

converges to $G^{(k)}(\zeta) - \zeta^m \neq 0$ as $\frac{1}{b(\rho_n \zeta)} = \frac{1}{b(0)} = 1$. Hence $G^{(k)}(\zeta) \equiv \zeta^m$. Then by Hurwitz's theorem G is a non constant polynomial. Therefore G has a zero at ζ_0 . Since all zeros of G have multiplicity at least $k + 1$, so we have $\zeta_0^m = G^{(k)}(\zeta_0) = 0$. Hence $\zeta_0 = 0$. This contradicts that $G(0) \neq 0$. Thus $G^{(k)}(\zeta) \neq \zeta^m$. It follows from Lemma 3 that $G(\zeta)$ is a rational function. However Lemma 4 shows that $G^{(k)}(\zeta) = \zeta^m$ has a solution in $\mathbb{C}$. This contradiction shows that $\mathcal{H}$ is normal at 0.

We shall prove that $\mathcal{F}$ is normal at 0.

Suppose $\{f_n(z)\}$ is a sequence in $\mathcal{F}$. Set $F_n(z) = \frac{f_n(z)}{z^m}$. Then $\{F_n(z)\} \subset \mathcal{H}$. Since $\mathcal{H}$ is normal at $z = 0$, then there exists $\Delta_\delta = \{z: |z| < \delta\}$ and a sub sequence of $\{F_n(z)\}$ such that $\{F_n(z)\}$ converges uniformly to a meromorphic function $F(z)$ or ∞ on Δ_δ . Note that $F(0) = \infty$, so we can find $\varepsilon \in [0, \delta]$ and a positive constant M such that $|F(z)| \geq M \quad \forall z \in \Delta_\varepsilon$. So sufficiently large n we have $|F_n(z)| \geq \frac{M}{2}$. Thus $f_n(z) \neq 0$ for sufficiently large n and $z \in \Delta_\varepsilon$. Therefore $\frac{1}{f_n}$ are analytic on Δ_ε . Thus for sufficiently large n ,

$$\left| \frac{1}{f_n(z)} \right| = \left| \frac{1}{F_n(z)} \cdot \frac{1}{|z^m|} \right| \leq \frac{2^m}{\delta^m} \cdot \frac{2}{M} \text{ as } |z| = \frac{\varepsilon}{2}.$$

So by maximum principle and Montel's theorem, $\mathcal{F}$ is normal at $z = 0$.

Proof of Theorem 2. Without loss of generality we assume that Ω is a disk and $a_0, a_1, \dots, a_{k-1}, a$ are bounded in Ω . Using the relation $h(z) = g(\alpha + \beta z)$ and $(f \circ h)^{(j)}(z) = \beta^j (f \circ g)^{(j)}(\alpha + \beta z)$ for a suitable choice of $\alpha, \beta \in \mathbb{C}$. Also we assume that Ω is $B(0, 1)$. Suppose $\mathcal{G}$ is not normal on Ω . Then $\mathcal{G}$ must not be normal at least one point in Ω . We may set at $z = 0$ $\mathcal{G}$ is not normal.

Since $\mathcal{G}$ is a family of analytic functions then by Lemma 6, there exists points $z_n \rightarrow 0$ and sequence of positive numbers $\rho_n \rightarrow 0$ and $\{g_n\}$ in $\mathcal{G}$ and a non constant entire function g such that

$$h_n(z) = g_n(z_n + \rho_n z) \rightarrow g(z)$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric with $g^\# \leq 1$. Then by Lemma 7, g is of finite order. For an entire function g , $(f \circ h_n)(z) \rightarrow (f \circ g)(z)$ locally uniformly on $\mathbb{C}$. Then by Weierstrass's Theorem we have

$$(f \circ h_n)^{(j)}(z) = \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \rightarrow (f \circ g)^{(j)}(z)$$

locally uniformly on $\mathbb{C}$. Since each $g_n \in \mathcal{G}$ and $(z_n + \rho_n z) \in \Omega$, so

$$\begin{aligned} F_n(z) &= (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} a_j(z_n + \rho_n z) (f \circ g_n)^{(j)}(z_n + \rho_n z) \\ &\quad - a(z_n + \rho_n z) (f \circ g_n)^n(z_n + \rho_n z) \neq b(z_n + \rho_n z). \end{aligned}$$

Therefore we have

$$\begin{aligned} \rho_n^k F_n(z) &= \rho_n^k (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} \rho_n^{k-j} a_j(z_n + \rho_n z) \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \\ &\quad - \rho_n^k a(z_n + \rho_n z) (f \circ g_n)^n(z_n + \rho_n z) \neq \rho_n^k b(z_n + \rho_n z). \end{aligned}$$

Since $\rho_n^{k-j} \rightarrow 0$ for $j = 0, 1, 2, \dots, k-1$ and a_j 's are assumed bounded on Ω , then

$$\lim_{n \rightarrow \infty} \rho_n^k F_n(z) = (f \circ g)^{(k)}(z)$$

locally uniformly on $\mathbb{C}$. Hence by Lemma 5, we have either $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$ or $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$.

Case I. Let $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$. Then we have $(f \circ g)'(z) \equiv P_{k-2}(z)$, a polynomial of degree at most $k-2$. Since $P_{k-2}(z)$ has at most $k-2$ zeros counting multiplicities then $(f \circ g)'(z) = f'(g(z))g'(z)$ has also the same zeros. Since f is a transcendental entire function with $\rho(f) < 1$ and so by Lemma 10, f' has infinitely many zeros on $\mathbb{C}$. Since g is a non constant entire function, we must have g omits infinitely many zeros of f' on $\mathbb{C}$, which is a contradiction by Picard's theorem.

Case II. Let $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$. Suppose $k = 1$. Then by Lemma 8, we have $(f \circ g)'(z)$ has infinitely many zeros on $\mathbb{C}$, a contradiction. Suppose $k \geq 2$, then since f is transcendental entire function with $\rho(f) < \frac{1}{k}$ and g is a non constant entire function with finite order, so by Lemma 9, $(f \circ g)^{(k)}$ has at least one zero in $\mathbb{C}$, a contradiction. So $\mathcal{G}$ is a normal family.

References

- [1] C. Lie, M. Fang, D. Yang, X. Wang. "Exceptional functions and normal families of Meromorphic functions with multiple zeros", J. Math. Anal. Appl. Vol. 341, pp. 224-234, 2008.
- [2] E.F. Clifford. "Normal families and value distribution in connection with composite functions", J. Math. Anal. Appl. Vol. 312, pp. 195-204, 2005.
- [3] W. K. Hayman. "Meromorphic functions", Clarendon Press, Oxford, 1964.
- [4] J. L. Schiff. "Normal Families", Springer Verlag, Berlin, 1993.
- [5] L. Yang. "Value distribution theory", Springer Verlag, Berlin, 1993.
- [6] Y. X. Gu. "A normal criterion of meromorphic families", Scientia Sinica. Math, Issue 1, pp. 267-274, 1979.
- [7] Y. F. Wang, M. L. Fang. "Picard values and normal families of Meromorphic functions with multiple zeros", Acta Math. Sinica (N. S), Vol. 14, pp. 17-26, 1998.
- [8] X. C. Pang, D.G. Yang, L. Zalcman. "Normal families of Meromorphic functions whose derivatives omit a function", Comput. Methods Funct. Theory, Vol. 2, pp. 257-265, 2002.
- [9] Y. Xu. "Normality and exceptional functions of derivatives", J. Aust.Math. Soc. Vol. 76, pp. 403-413, 2004.
- [10] J. D. Hincliffe. "Normality and fixpoints of analytic functions", Proc. Roy. Soc. Edinburgh Sect. Vol. A 133 pp. 1335-1339, 2003.
- [11] X. C. Pang, L. Zalcman. "Normal families shared values", Bull. London Math. Soc. Vol. 32, pp. 325-331, 2000.
- [12] L. Zalcman. "Normal Families: New Perspectives", Bull. Amer. Math. Soc.(N. S), Vol. 35, pp. 215-230, 1998.
- [13] X. C. Pang, L. Zalcman. "Normal families of Meromorphic functions with multiple zeros and poles", Israel J. Math, Vol. 136, pp. 1-9, 2003.

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