



An approach to generalized extrapolation formula based on rate of changes (derivatives).

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Abstract: We are proposing a generalized form of extrapolation formula for approximating the values of a dependent variable under five initial conditions (arguments) based on rate of changes of that variable corresponding to an independent variable (derivative of dependent variable with respect to independent variable). As the approximate values of dependent variable may fluctuate from the exact solution values for truncation & round-off errors, we are trying to minimize the fluctuation by adding proper correcting factor. Here correcting factors will be obtained from fifth order finite difference [6,7] of the first one of given initial values. Also, we will compare approximate values with exact solution values by highlighting relative errors (expressed in percentage).

Keywords: extrapolation, finite difference, correcting factor, quantitative comparison, relative error.

I. Introduction

Extrapolation is the process of estimating, beyond the original observation interval, the value of a variable on the basis of its relationship with another variable. Extrapolation is subject to greater uncertainty and a higher risk of producing meaningless results. Extrapolation may also mean extension of a method, assuming similar methods will be applicable. Extrapolation may also apply to human experience to project, extend, or expand known experience into an area not known or previously experienced so as to arrive at a (usually conjectural) knowledge of the unknown [4] (e.g. a driver extrapolates road conditions beyond his sight while driving). Linear extrapolation means creating a tangent line at the end of the known data and extending it beyond that limit. Linear extrapolation will only provide good results when used to extend the graph of an approximately linear function or not too far beyond the known data.

A sound choice of which extrapolation method to apply relies on a prior knowledge of the process that created the existing data points. Some experts have proposed the use of causal forces in the evaluation of extrapolation methods [5]. Crucial questions are for example if the data can be assumed to be continuous, smooth, possibly periodic etc.

If the two data points nearest the point x_r to be extrapolated are (x_{k-1}, y_{k-1}) and (x_k, y_k) linear extrapolation [7] gives the function:

$$y(x_r) = y_{k-1} + \frac{x_r - x_{k-1}}{x_k - x_{k-1}} (y_k - y_{k-1})$$

... .. (1)

Extrapolation attempts to predict future data by relying on historical data, such as estimating the size of a population a few years from now on the basis of current population size and its rate of growth. It may be valid where the present circumstances do not indicate any interruption in the long-established past trends. However, a straight line extrapolation (where a short-term trend is believed to continue far into future) is fraught with risk because some unforeseeable factors almost always intervene.

II. Derivation of generalized extrapolation formula

We know that Newton's formula of forward interpolation [3,7] in terms of y' and u is given by

Now, integrating (2) over the interval $x_0 + th$ to $x_0 + rh$ i.e., $u = t$ to $u = r$, we obtain

$$\int_{x_0+th}^{x_0+rh} y' dx = h \int_t^r [y'_0 + u \Delta y'_0 + \frac{(u^2 - u)}{2} \Delta^2 y'_0 + \frac{(u^3 - 3u^2 + 2u)}{6} \Delta^3 y'_0 + \frac{(u^4 - 6u^3 + 11u^2 - 6u)}{24} \Delta^4 y'_0 + \frac{(u^5 - 10u^4 + 35u^3 - 50u^2 + 24u)}{120} \Delta^5 y'_0 + \dots \dots \dots] du$$

$$\text{or, } [y]_{x_0+th}^{x_0+rh} = h[uy'_0 + \frac{u^2}{2}\Delta y'_0 + \frac{1}{2}(\frac{u^3}{3} - \frac{u^2}{2})\Delta^2 y'_0 + \frac{1}{6}(\frac{u^4}{4} - u^3 + u^2)\Delta^3 y'_0 \\ + \frac{1}{24}(\frac{u^5}{5} - \frac{3u^4}{2} + \frac{11u^3}{3} - 3u^2)\Delta^4 y'_0 + \frac{1}{120}(\frac{u^6}{6} - 2u^5 + \frac{35u^4}{4} - \frac{50u^3}{3} + 12u^2)\Delta^5 y'_0 + \dots\dots\dots]_{u=t}^{u=r}$$

$$\begin{aligned} \text{or, } y(x_0 + rh) - y(x_0 + th) = & h[(r-t)y'_0 + (\frac{r^2}{2} - \frac{t^2}{2})\Delta y'_0 + (\frac{r^3}{6} - \frac{r^2}{4} - \frac{t^3}{6} + \frac{t^2}{4})\Delta^2 y'_0 \\ & + (\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{t^4}{24} + \frac{t^3}{6} - \frac{t^2}{6})\Delta^3 y'_0 + (\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} - \frac{t^5}{120} + \frac{t^4}{16} - \frac{11t^3}{72} + \frac{t^2}{8})\Delta^4 y'_0 \\ & + (\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} - \frac{t^6}{720} + \frac{t^5}{60} - \frac{7t^4}{96} + \frac{5t^3}{36} - \frac{t^2}{10})\Delta^5 y'_0 + \dots\dots\dots] \end{aligned}$$

$$\begin{aligned} \text{or, } y_r - y_t &= h[(r-t)y'_0 + (\frac{r^2}{2} - \frac{t^2}{2})\Delta y'_0 + (\frac{r^3}{6} - \frac{r^2}{4} - \frac{t^3}{6} + \frac{t^2}{4})\Delta^2 y'_0 \\ &+ (\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{t^4}{24} + \frac{t^3}{6} - \frac{t^2}{6})\Delta^3 y'_0 + (\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} - \frac{t^5}{120} + \frac{t^4}{16} - \frac{11t^3}{72} + \frac{t^2}{8})\Delta^4 y'_0 \\ &+ (\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} - \frac{t^6}{720} + \frac{t^5}{60} - \frac{7t^4}{96} + \frac{5t^3}{36} - \frac{t^2}{10})\Delta^5 y'_0 + \dots\dots\dots] \end{aligned}$$

Neglecting the terms containing Δ^6 and higher orders and substituting $\Delta \equiv E - 1$, from (3) we get as followings

$$y_r - y_t = h[(r-t)y'_0 + (\frac{r^2}{2} - \frac{t^2}{2})(E-1)y'_0 + (\frac{r^3}{6} - \frac{r^2}{4} - \frac{t^3}{6} + \frac{t^2}{4})(E-1)^2y'_0 + (\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{t^4}{24} + \frac{t^3}{6} - \frac{t^2}{6})(E-1)^3y'_0$$

$$\begin{aligned}
 & + \left(\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} - \frac{t^5}{120} + \frac{t^4}{16} - \frac{11t^3}{72} + \frac{t^2}{8} \right) (E-1)^4 y'_0 \\
 & + \left(\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} - \frac{t^6}{720} + \frac{t^5}{60} - \frac{7t^4}{96} + \frac{5t^3}{36} - \frac{t^2}{10} \right) \Delta^5 y'_0] \\
 \text{or, } y_r - y_t &= h \left[(r-t)y'_0 + \left(\frac{r^2}{2} - \frac{t^2}{2} \right) (E-1)y'_0 + \left(\frac{r^3}{6} - \frac{r^2}{4} - \frac{t^3}{6} + \frac{t^2}{4} \right) (E^2 - 2E + 1)y'_0 \right. \\
 & + \left(\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{t^4}{24} + \frac{t^3}{6} - \frac{t^2}{6} \right) (E^3 - 3E^2 + 3E - 1)y'_0 \\
 & + \left(\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} - \frac{t^5}{120} + \frac{t^4}{16} - \frac{11t^3}{72} + \frac{t^2}{8} \right) (E^4 - 4E^3 + 6E^2 - 4E + 1)y'_0 \\
 & \left. + \left(\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} - \frac{t^6}{720} + \frac{t^5}{60} - \frac{7t^4}{96} + \frac{5t^3}{36} - \frac{t^2}{10} \right) \Delta^5 y'_0 \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{or, } y_r &= y_t + h \left[(r-t)y'_0 + \left(\frac{r^2}{2} - \frac{t^2}{2} \right) (y'_1 - y'_0) + \left(\frac{r^3}{6} - \frac{r^2}{4} - \frac{t^3}{6} + \frac{t^2}{4} \right) (y'_2 - 2y'_1 + y'_0) \right. \\
 & + \left(\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{t^4}{24} + \frac{t^3}{6} - \frac{t^2}{6} \right) (y'_3 - 3y'_2 + 3y'_1 - y'_0) \\
 & + \left(\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} - \frac{t^5}{120} + \frac{t^4}{16} - \frac{11t^3}{72} + \frac{t^2}{8} \right) (y'_4 - 4y'_3 + 6y'_2 - 4y'_1 + y'_0) \\
 & \left. + \left(\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} - \frac{t^6}{720} + \frac{t^5}{60} - \frac{7t^4}{96} + \frac{5t^3}{36} - \frac{t^2}{10} \right) \Delta^5 y'_0 \right]
 \end{aligned}$$

$$\text{[Since } Ey'_0 = y'_1, E^2 y'_0 = y'_2, E^3 y'_0 = y'_3, E^4 y'_0 = y'_4]$$

$$\begin{aligned}
 \text{or, } y_r &= y_t + h \left[\left\{ \left(r - \frac{r^2}{2} + \frac{r^3}{6} - \frac{r^2}{4} - \frac{r^4}{24} + \frac{r^3}{6} - \frac{r^2}{6} + \frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} \right) - \left(t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^2}{4} - \frac{t^4}{24} + \frac{t^3}{6} \right. \right. \right. \\
 & \left. \left. - \frac{t^2}{6} + \frac{t^5}{120} - \frac{t^4}{16} + \frac{11t^3}{72} - \frac{t^2}{8} \right) \right\} y'_0 \\
 & + \left\{ \left(\frac{r^2}{2} - \frac{r^3}{3} + \frac{r^2}{2} + \frac{r^4}{8} - \frac{r^3}{2} + \frac{r^2}{2} - \frac{r^5}{30} + \frac{r^4}{4} - \frac{11r^3}{18} + \frac{r^2}{2} \right) - \left(\frac{t^2}{2} - \frac{t^3}{3} + \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^3}{2} + \frac{t^2}{2} - \frac{t^5}{30} + \frac{t^4}{4} - \frac{11t^3}{18} \right. \right. \\
 & \left. \left. + \frac{t^2}{2} \right) \right\} y'_1 \\
 & + \left\{ \left(\frac{r^3}{6} - \frac{r^2}{4} - \frac{r^4}{8} + \frac{r^3}{2} - \frac{r^2}{2} + \frac{r^5}{20} - \frac{3r^4}{8} + \frac{11r^3}{12} - \frac{3r^2}{4} \right) - \left(\frac{t^3}{6} - \frac{t^2}{4} - \frac{t^4}{8} + \frac{t^3}{2} - \frac{t^2}{2} + \frac{t^5}{20} - \frac{3t^4}{8} + \frac{11t^3}{12} - \frac{3t^2}{4} \right) \right\} y'_2 \\
 & + \left\{ \left(\frac{r^4}{24} - \frac{r^3}{6} + \frac{r^2}{6} - \frac{r^5}{30} + \frac{r^4}{4} - \frac{11r^3}{18} + \frac{r^2}{2} \right) - \left(\frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{6} - \frac{t^5}{30} + \frac{t^4}{4} - \frac{11t^3}{18} + \frac{t^2}{2} \right) \right\} y'_3 \\
 & + \left\{ \left(\frac{r^5}{120} - \frac{r^4}{16} + \frac{11r^3}{72} - \frac{r^2}{8} \right) - \left(\frac{t^5}{120} - \frac{t^4}{16} + \frac{11t^3}{72} - \frac{t^2}{8} \right) \right\} y'_4 \right] \\
 & + \left\{ \left(\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} \right) - \left(\frac{t^6}{720} - \frac{t^5}{60} + \frac{7t^4}{96} - \frac{5t^3}{36} + \frac{t^2}{10} \right) \right\} h \Delta^5 y'_0 \\
 \text{or, } y_r &= y_t + h \left[\left\{ \left(r - \frac{25r^2}{24} + \frac{35r^3}{72} - \frac{5r^4}{48} + \frac{r^5}{120} \right) - \left(t - \frac{25t^2}{24} + \frac{35t^3}{72} - \frac{5t^4}{48} + \frac{t^5}{120} \right) \right\} y'_0 \right. \\
 & + \left\{ \left(2r^2 - \frac{13r^3}{9} + \frac{3r^4}{8} - \frac{r^5}{30} \right) - \left(2t^2 - \frac{13t^3}{9} + \frac{3t^4}{8} - \frac{t^5}{30} \right) \right\} y'_1 \\
 & \left. + \left\{ \left(-\frac{3r^2}{2} + \frac{19r^3}{12} - \frac{r^4}{2} + \frac{r^5}{20} \right) - \left(-\frac{3t^2}{2} + \frac{19t^3}{12} - \frac{t^4}{2} + \frac{t^5}{20} \right) \right\} y'_2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \left(\frac{2r^2}{3} - \frac{7r^3}{9} + \frac{7r^4}{24} - \frac{r^5}{30} \right) - \left(\frac{2t^2}{3} - \frac{7t^3}{9} + \frac{7t^4}{24} - \frac{t^5}{30} \right) \right\} y_3' \\
 & \quad + \left\{ \left(-\frac{r^2}{8} + \frac{11r^3}{72} - \frac{r^4}{16} + \frac{r^5}{120} \right) - \left(-\frac{t^2}{8} + \frac{11t^3}{72} - \frac{t^4}{16} + \frac{t^5}{120} \right) \right\} y_4' \\
 & + \left\{ \left(\frac{r^6}{720} - \frac{r^5}{60} + \frac{7r^4}{96} - \frac{5r^3}{36} + \frac{r^2}{10} \right) - \left(\frac{t^6}{720} - \frac{t^5}{60} + \frac{7t^4}{96} - \frac{5t^3}{36} + \frac{t^2}{10} \right) \right\} h \Delta^5 y_0' \\
 \text{or, } y_r = y_t & + \frac{h}{1440} \left[\{ (1440r - 1500r^2 + 700r^3 - 150r^4 + 12r^5) - (1440t - 1500t^2 + 700t^3 - 150t^4 \right. \\
 & \quad \left. + 12t^5) \} y_0' \right. \\
 & + \{ (2880r^2 - 2080r^3 + 540r^4 - 48r^5) - (2880t^2 - 2080t^3 + 540t^4 - 48t^5) \} y_1' \\
 & + \{ (-2160r^2 + 2280r^3 - 720r^4 + 72r^5) - (-2160t^2 + 2280t^3 - 720t^4 + 72t^5) \} y_2' \\
 & + \{ (960r^2 - 1120r^3 + 420r^4 - 48r^5) - (960t^2 - 1120t^3 + 420t^4 - 48t^5) \} y_3' \\
 & + \{ (-180r^2 + 220r^3 - 90r^4 + 12r^5) - (-180t^2 + 220t^3 - 90t^4 + 12t^5) \} y_4' \\
 & + \{ (144r^2 - 200r^3 + 105r^4 - 24r^5 + 2r^6) - (144t^2 - 200t^3 + 105t^4 - 24t^5 + 2t^6) \} \frac{h \Delta^5 y_0'}{1440} \\
 & \dots \dots \dots (4)
 \end{aligned}$$

Now, we can put $t = 0, 1, 2, 3, 4$ in equation (4), for better approximation putting $t = 1$, then we get as follows

$$\begin{aligned}
 y_r = y_1 & + \frac{h}{1440} \left[\{ (1440r - 1500r^2 + 700r^3 - 150r^4 + 12r^5) - (1440 - 1500 + 700 - 150 + 12) \} y_0' \right. \\
 & \quad + \{ (2880r^2 - 2080r^3 + 540r^4 - 48r^5) - (2880 - 2080 + 540 - 48) \} y_1' \\
 & \quad \left. + \{ (-2160r^2 + 2280r^3 - 720r^4 + 72r^5) - (-2160 + 2280 - 720 + 72) \} y_2' \right. \\
 & \quad + \{ (960r^2 - 1120r^3 + 420r^4 - 48r^5) - (960 - 1120 + 420 - 48) \} y_3' \\
 & \quad \left. + \{ (-180r^2 + 220r^3 - 90r^4 + 12r^5) - (-180 + 220 - 90 + 12) \} y_4' \right] \\
 & + \{ (144r^2 - 200r^3 + 105r^4 - 24r^5 + 2r^6) - (144 - 200 + 105 - 24 + 2) \} \frac{h \Delta^5 y_0'}{1440} \\
 \text{or, } y_r = y_1 & + \frac{h}{1440} \left[(1440r - 1500r^2 + 700r^3 - 150r^4 + 12r^5 - 502) y_0' \right. \\
 & + (2880r^2 - 2080r^3 + 540r^4 - 48r^5 - 1292) y_1' + (-2160r^2 + 2280r^3 - 720r^4 + 72r^5 + 528) y_2' \\
 & + (960r^2 - 1120r^3 + 420r^4 - 48r^5 - 212) y_3' + (-180r^2 + 220r^3 - 90r^4 + 12r^5 + 38) y_4' \\
 & \quad \left. + (144r^2 - 200r^3 + 105r^4 - 24r^5 + 2r^6 - 27) \frac{h \Delta^5 y_0'}{1440} \right. \\
 & \dots \dots \dots (5)
 \end{aligned}$$

Thus, the above formula takes the form as

$$\begin{aligned}
 y_r = y_1 & + \frac{h}{1440} \left[(1440r - 1500r^2 + 700r^3 - 150r^4 + 12r^5 - 502) y_0' \right. \\
 & + (2880r^2 - 2080r^3 + 540r^4 - 48r^5 - 1292) y_1' + (-2160r^2 + 2280r^3 - 720r^4 + 72r^5 + 528) y_2' \\
 & \left. + (960r^2 - 1120r^3 + 420r^4 - 48r^5 - 212) y_3' + (-180r^2 + 220r^3 - 90r^4 + 12r^5 + 38) y_4' \right]
 \end{aligned}$$

$$\dots \dots \dots (6)$$

Here the correcting factor is taking as

$$CF_r = (144r^2 - 200r^3 + 105r^4 - 24r^5 + 2r^6 - 27) \frac{h\Delta^5 y'_0}{1440}$$

$$\dots \dots \dots (7)$$

III. Particular forms

Finally, for particular use putting $r = 5, 7, 10, 12, 15$ in (6) and (7), then we obtain

$$y_5 = y_1 + \frac{h}{1440} [448y'_0 - 1792y'_1 + 6528y'_2 - 3712y'_3 + 4288y'_4]$$

$$y_7 = y_1 + \frac{h}{1440} [17712y'_0 - 83808y'_1 + 158112y'_2 - 135648y'_3 + 52272y'_4]$$

$$y_{10} = y_1 + \frac{h}{1440} [263898y'_0 - 1193292y'_1 + 2064528y'_2 - 1624212y'_3 + 502038y'_4]$$

$$y_{12} = y_1 + \frac{h}{1440} [885962y'_0 - 3927308y'_1 + 6615312y'_2 - 5032148y'_3 + 1474022y'_4]$$

$$y_{15} = y_1 + \frac{h}{1440} [3564848y'_0 - 15485792y'_1 + 25434528y'_2 - 18751712y'_3 + 5258288y'_4]$$

With the corrector factors as

$$CF_5 = \frac{448h\Delta^5 y'_0}{1440} ; \quad CF_7 = \frac{22464h\Delta^5 y'_0}{1440} ; \quad CF_{10} = \frac{464373h\Delta^5 y'_0}{1440} ; \quad CF_{12} = \frac{1852389h\Delta^5 y'_0}{1440} ;$$

$$CF_{15} = \frac{9229248h\Delta^5 y'_0}{1440}$$

Here for corrected approximate values of $y(x)$, we have to add CF_r with y_r . Also for y_5 fifth finite difference will choose as zero and with y_5 fifth finite difference for other approximate values to be chosen.

$$i.e. y_r^c = y_r + CF_r$$

Above formula is the required generalized extrapolation formula.

IV. Numerical examples

Problem-1: Solve $y' = 2e^x - y$ for the values of x at 0.50, 0.70, 1.00, 1.20 and 1.50

With the initial values [1,8] given as $y(0.00) = 2.0000, y(0.10) = 2.0100,$
 $y(0.20) = 2.0401, y(0.30) = 2.0907, y(0.40) = 2.1621$

The analytical solution is $Y = e^x + e^{-x}$

Problem-2: Solve $y' = 3e^x + 2y$ for the values of x at 0.50, 0.70, 1.00, 1.20 and 1.50

With the initial values [1,2,8] given as $y(0.00) = 0.0000, y(0.10) = 0.3487,$
 $y(0.20) = 0.8113, y(0.30) = 1.4168, y(0.40) = 2.2012$

The analytical solution is $Y = 3(e^{2x} - e^x)$

Problem-3: Solve $y' = x^2 + y$ for the values of x at 1.25, 1.75, 2.50, 3.00 and 3.75

With the initial values [8] given as $y(0.00) = 0.0000, y(0.25) = 0.0056,$
 $y(0.50) = 0.0474, y(0.75) = 0.1715, y(1.00) = 0.4366$

The analytical solution is $Y = 2(e^x - x - 1) - x^2$

Problem-04: Solve $y' = e^x + y$ for the values of x at 1.25, 1.75, 2.50, 3.00 and 3.75

With the initial values [8] given as $y(0.00) = 1.0000, y(0.25) = 1.6050,$
 $y(0.50) = 2.4731, y(0.75) = 3.7048, y(1.00) = 5.4366$

The analytical solution is $Y = e^x(x + 1)$

Problem-05: Solve $y' = \frac{(x+y)}{2}$ for the values of x at 2.50, 3.50, 5.00, 6.00 and 7.50

With the initial values [1] given as $y(0.00) = 2.0000, y(0.50) = 2.6361,$
 $y(1.00) = 3.5949, y(1.50) = 4.9680, y(2.00) = 6.8731$

The analytical solution is $Y = 4e^{\frac{x}{2}} - x - 2$

V. Quantitative comparison of numerical results

Problem	r	h	x	y_r	Δ^5	CF_r	y_r^c	Y_r	$ E_R $
01	05	0.10	0.50	2.2553	0.0003	0.0000	2.2553	2.2553	0.000%
	07		0.70	2.5112		0.0005	2.5117	2.5103	0.056%
	10		1.00	3.0967		0.0097	3.1064	3.0862	0.655%
	12		1.20	3.6543		0.0386	3.6929	3.6213	1.977%
	15		1.50	4.8275		0.1923	5.0198	4.7048	6.695%

Problem	r	h	x	y_r	Δ^5	CF_r	y_r^c	Y_r	$ E_R $
02	05	0.10	0.50	3.2086	0.0011	0.0000	3.2086	3.2087	0.003%
	07		0.70	6.1197		0.0017	6.1214	6.1243	0.047%
	10		1.00	13.9005		0.0355	13.9360	14.0123	0.545%
	12		1.20	22.6238		0.1415	22.7653	23.1092	1.488%
	15		1.50	44.6739		0.7050	45.3789	46.8115	3.060%

Problem	r	h	x	y_r	Δ^5	CF_r	y_r^c	Y_r	$ E_R $
03	05	0.25	1.25	0.9179	0.0039	0.0000	0.9179	0.9182	0.033%
	07		1.75	2.9298		0.0152	2.9450	2.9467	0.058%
	10		2.50	10.7319		0.3144	11.0463	11.1150	0.618%
	12		3.00	21.5208		1.2542	22.7750	23.1711	1.710%
	15		3.75	52.0796		6.2490	58.3286	61.4797	5.125%

Problem	r	h	x	y_r	Δ^5	CF_r	y_r^c	Y_r	$ E_R $
04	05	0.25	1.25	7.8522	0.0112	0.0000	7.8522	7.8533	0.014%
	07		1.75	15.7658		0.0437	15.8095	15.8252	0.099%
	10		2.50	41.2194		0.9030	42.1224	42.6387	1.211%
	12		3.00	74.0472		3.6019	77.5491	80.3422	3.352%
	15		3.75	164.6912		17.9458	182.6370	201.9751	9.575%

Problem	r	h	x	y_r	Δ^5	CF_r	y_r^c	Y_r	$ E_R $
05	05	0.50	2.50	9.4610	0.0070	0.0000	9.4610	9.4614	0.004%
	07		3.50	17.4900		0.0546	17.5446	17.5184	0.150%
	10		5.00	41.0354		1.1287	42.1641	41.7300	1.040%
	12		6.00	69.2712		4.5023	73.7735	72.3422	1.979%
	15		7.50	142.6672		22.4322	165.0994	160.5843	2.812%

VI. Conclusions

From above comparison table we have shown that proposed generalized extrapolation formula gives very good accuracy upto 10th approximate values of dependent variable. Also, 12th approximate value of dependent variable is in a good accuracy. But the next approximate values of dependent variable fluctuated much. The approximate values of dependent variable may fluctuate from the exact solution values for truncation & round-off errors. Also, this fluctuation occurred for taking CF_5 is equal to zero. We will try to overcome this problem of fluctuation at next. Moreover, we have seen that for the smaller h (step size) the proposed formula gives better accuracy and fluctuation increases simultaneously with h. So, we can conclude that generalized extrapolation formula is efficient to obtain the approximate values of dependent variable for smaller h and for any h it will provide a good accuracy at least upto 12th approximations. Further, we will try to implement generalized extrapolation formula in real world problems.

References

- [1] A. R. VASISTHA, V. VASISTHA, "Numerical analysis", KedarNath-Ram Nath, Meerut, 1999, PP.284, 288.
- [2] B. D. SHARMA, "Differential equations", KedarNath-Ram Nath, Meerut, 2006, PP.159
- [3] FRANCIS SCHELD, "Numerical analysis", Schaum's Outline Series, McGraw-Hill, 1988, PP.471.

- [3] C. Brezinski and M. Redivo Zaglia "Extrapolation Methods. Theory and Practice", North-Holland, 1991.
- [4] J. Scott Armstrong and Fred Collopy, "Causal Forces: Structuring Knowledge for Time-series Extrapolation", Journal of Forecasting-12, PP.103–115.
- [5] JAMES B. SCARBOROUGH, "Numerical mathematical analysis" Oxford and IBM Publishing Co. Pvt. Ltd., 1966, PP.48.
- [6] S. BALACHANDRA RAO and C. K. SHANTHA, "Numerical methods", Universities Press India Ltd., Hyderabad, 2000, PP.86, 121, 125.
- [7] V. N. VEDAMURTHY and N. Ch. S. N. IYENGAR, "Numerical methods", Vikas Publishing house Private Ltd., New Delhi, 2002, PP.11.21, 11.32, 11.58.