



Differential Subordination Results For Meromorphic Multivalent Functions Applying A Linear Operator Associated with Generalized Hypergeometric Functions

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Abstract: Our purpose in this paper is to define a linear operator $D_{p,\lambda}^n$, and taking the advantage of the Hadamard product (or convolution) to associate it with the generalized hypergeometric function, obtaining the operator $L_{p,\lambda,q,s}^n$ and then using this operator to get some subordination results for these meromorphic multivalent functions belonging to the subclasses $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$ and $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$.

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1. INTRODUCTION

Let $A = A(U)$ denote the class of all analytic functions in the open unit disc

$U = \{z \in \mathbb{C} : |z| < 1\}$.

Consider

$$\Omega = \{w \in A : w(0) = 0 \text{ and } |w(z)| < 1\} \quad (z \in U), \quad (1.1)$$

the class of Schwarz functions.

For $0 \leq \alpha < 1$, let

$$\mathbf{P}(\alpha) = \{p \in A : p(0) = 1 \text{ and } \operatorname{Re}(p(z)) > \alpha\} \quad (z \in U). \quad (1.2)$$

Note that $\mathbf{P} = \mathbf{P}(0)$ is the well-known Carathéodory class of functions.

The classes of Schwarz and Carathéodory functions play an extremely important role in the theory of analytic functions and have been studied by many authors.

It is easy to see that

$$p \in \mathbf{P}(\alpha) \text{ if and only if } \frac{p(z) - \alpha}{1 - \alpha} \in \mathbf{P}(\alpha). \quad (1.3)$$

The following lemmas are needed for proving our results

Lemma 1.1 [10] Let $p \in A$. Then $p \in \mathbf{P}(\alpha)$ if and only if there exists $w \in \Omega$ such that

$$p(z) = \frac{1 - (2\alpha - 1)w(z)}{1 - w(z)} \quad (z \in U). \quad (1.4)$$

Lemma 1.2 [10] (Herglotz formula) A function $p \in A$ belongs to the class $\mathbf{P}$ if and only if there exists a probability measure $\mu(x)$ on ∂U , such that

$$p(z) = \int_{|x|=1} \frac{1 - (2\alpha - 1)xz}{1 - xz} d\mu(x) \quad (z \in U). \quad (1.5)$$

Let f and g be functions in A . The function f is said to be subordinate to g , or g is said to be superordinate to f if there exists a function $w \in \Omega$, such that $f(z) = g(w(z))$.

In such a case, we write

$$f < g \text{ or } f(z) < g(z) \quad (z \in U).$$

If the function g is univalent in U , then we have (see [9])

$$f(z) < g(z) \quad (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let Σ_p denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{k=1+p}^{\infty} a_k z^k \quad (p \in \mathbf{N} = \{1, 2, 3, \dots\}), \quad (1.6)$$

which are analytic and p -valent in the punctured unit disc $U^* = U \setminus \{0\}$.

Denote by Σ_p^+ the subclass of Σ_p consisting of functions of the form

$$f(z) = z^{-p} + \sum_{k=1+p}^{\infty} a_k z^k, \quad a_k \geq 0 \quad (z \in U^*). \quad (1.7)$$

A function $f \in \Sigma_p$ is meromorphically multivalent starlike of order α ($0 \leq \alpha < 1$) see [1] if

$$-\operatorname{Re}\left\{\frac{1}{p} \frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U).$$

The class of all such functions is denoted $\Sigma_p^*(\alpha)$.

If $f \in \Sigma_p$ is given by (1.6) and $g \in \Sigma_p$ is given by

$$g(z) = z^{-p} + \sum_{k=1+p}^{\infty} b_k z^k \quad (z \in U^*),$$

then, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1+p}^{\infty} a_k b_k z^k = (g * f)(z), \quad (p \in \mathbf{N}, z \in U^*).$$

Now, for functions $f(z) \in \Sigma_p$ in the form (1.6) we define the linear operator

$$D_{p,\lambda}^n: \Sigma_p \rightarrow \Sigma_p \text{ as follows}$$

$$D_{p,\lambda}^0 f(z) = f(z) \text{ and } D_{p,\lambda}^n f(z) = f(z) \quad (0 \leq \lambda < p; p \in \mathbf{N}, n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}; z \in U^*),$$

$$D_{p,\lambda}^1 f(z) = D_{p,\lambda} f(z) = (1 + \lambda p^2) f(z) + \lambda p z f'(z),$$

or

$$D_{p,\lambda} f(z) = z^{-p} + \sum_{k=1+p}^{\infty} (1 + \lambda p^2 + \lambda p k) a_k z^k,$$

therefore

$$D_{p,\lambda}^n f(z) = z^{-p} + \sum_{k=1+p}^{\infty} (1 + \lambda p^2 + \lambda p k)^n a_k z^k \quad (1.8)$$

$$D_{p,\lambda}^n f(z) = z^{-p} + \sum_{k=1+p}^{\infty} \delta_k(\lambda, p, n) a_k z^k \quad (1.9)$$

where

$$\delta_k(\lambda, p, n) = (1 + \lambda p^2 + \lambda p k)^n.$$

By simple calculations and using (1.9), we can easily verify that

$$\lambda p z [D_{p,\lambda}^n f(z)]' = D_{p,\lambda}^{n+1} f(z) - (1 + \lambda p^2) D_{p,\lambda}^n f(z) \quad (1.10)$$

Following the same manner as in H. Orhan et al. [10], we use the linear operator $D_{p,\lambda}^n$ to define a subclass of the class Σ_p as follows

Definition 1.3. A function $f \in \Sigma_p$ is said to be in the subclass $D_{p,\lambda}^n(\alpha, \gamma)$ if it satisfies the condition

$$\left| \frac{1}{p} \frac{z(D_{p,\lambda}^n f(z))'}{D_{p,\lambda}^n f(z)} + 1 \right| < \gamma \quad \left| \frac{1}{p} \frac{z(D_{p,\lambda}^n f(z))'}{D_{p,\lambda}^n f(z)} + 2\alpha - 1 \right|, \quad (1.11)$$

for some α ($0 \leq \alpha < 1$), γ ($0 < \gamma \leq 1$) and ($z \in U^*$).

For parameters $\alpha_j \in \mathbf{C}$, ($j=1, \dots, q$) and $\beta_j \in \mathbf{C} \setminus \{0, -1, -2, \dots\}$ ($j=1, \dots, s$), the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is defined by the following infinite series (see [2,3,6]):

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \frac{z^k}{k!} \quad (q \leq s+1; q, s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}; z \in U),$$

where $(\theta)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\theta)_k = \frac{\Gamma(\theta+k)}{\Gamma(\theta)} = \begin{cases} 1 & \text{for } k = 0 \\ \theta(\theta+1)(\theta+2) \dots (\theta+k-1) & \text{for } k \in \mathbf{N} = \{1, 2, 3, \dots\}. \end{cases}$$

Corresponding to a function $h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z),$$

Liu-Srivastava [8] defined the operator $H_p^{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s); \Sigma_p \rightarrow \Sigma_p$ by the following Hadamard product (or convolution)

$$H_p^{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z) = h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z).$$

We also note that the definition of the operator $H_p^{q,s}(\alpha_1) = H_p^{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)$ was motivated essentially by Dziok and Srivastava [2]. Some interesting developments involving the Dziok-Srivastava operator were considered by (for example) Dziok and Srivastava [4], also see [6,7,11,13].

For functions belonging to Σ_p and satisfying the subclass $D_{p,\lambda}^n(\alpha, \gamma)$, we define the linear operator $L_{p,\lambda,q,s}^n$ as the Hadamard product (or convolution) of the operator $H_p^{q,s}(\alpha_1)$ with elements of the subclass $D_{p,\lambda}^n(\alpha, \gamma)$ as follows

$$\begin{aligned} L_{p,\lambda,q,s}^n f(z) &= H_p^{q,s}(\alpha_1) * D_{p,\lambda}^n f(z) \\ &= z^{-p} + \sum_{k=1+p}^{\infty} \mathcal{U} \sigma_{k,p}(\alpha_1) \delta_k(\lambda, p, n) a_k z^k \end{aligned} \quad (1.12)$$

where

$$\mathcal{U} = \frac{\prod_{j=1}^s \Gamma(\beta_j)}{\prod_{j=1}^q \Gamma(\alpha_j)} \quad \text{and} \quad \sigma_{k,p}(\alpha_1) = \frac{\prod_{j=1}^q \Gamma(\alpha_j + k - p)}{\prod_{j=1}^s \Gamma(\beta_j + k - p)}.$$

or

$$L_{p,\lambda,q,s}^n f(z) = z^{-p + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, \lambda, p, n)} a_k z^k, \quad (1.13)$$

where

$$\rho_k(\alpha_1, \lambda, p, n) = \mathcal{U} \sigma_{k,p}(\alpha_1) \delta_k(\lambda, p, n).$$

From (1.13), follows that $L_{p,\lambda,q,s}^n f(z)$ can be written in the convolution form as

$$L_{p,\lambda,q,s}^n f(z) = (f * l)(z), \quad (1.14)$$

where

$$l(z) = z^{-p + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, \lambda, p, n)} z^k. \quad (1.15)$$

Again following the manner of H. Orhan et al. [10], we use the linear operator $L_{p,\lambda,q,s}^n$ to define a subclass of the class Σ_p as follows

Definition 1.4. A function $f \in \Sigma_p$ is said to be in the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$ if it satisfies the condition

$$\left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1 \right| < \gamma \left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 2\alpha - 1 \right|, \quad (1.16)$$

for some, $\alpha_1 > 0$, α ($0 \leq \alpha < 1 < 1$), γ ($0 < \gamma \leq 1$) and ($z \in U^*$).

We consider another subclass of Σ_p given by

$$L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma] = \Sigma_p^+ \cap L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]. \quad (1.17)$$

In this paper a symmetric investigation of the subclasses $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$ and $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$ is presented.

2. PROPERTIES of THE SUBCLASS $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$

We start with the following theorem, in terms of subordination, for a function to be in the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$.

Theorem 2.1. A function $f \in \Sigma_p$ is in the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$ if and only if

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} < \frac{p(2\alpha-1)\gamma z - p}{1-\gamma z} \quad (z \in U). \quad (2.1)$$

Proof:

Let $f \in L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$. Then from Definition 1.4 we have

$$\left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1 \right| < \gamma \left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 2\alpha - 1 \right|,$$

or

$$\left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} - 1 \right|^2 < \gamma^2 \left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1 - 2\alpha \right|^2$$

or

$$\left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right|^2 - 2 \left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right| + 1 < \gamma^2 \left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right|^2 + 2\gamma^2(1-2\alpha) \left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right| + \gamma^2(1-2\alpha)^2$$

or

$$(1-\gamma^2) \left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right|^2 - 2[1+\gamma^2(1-2\alpha)] \operatorname{Re} \left\{ -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right\} < \gamma^2(1-2\alpha)^2 - 1$$

if $\gamma \neq 0$, we have

$$\left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right|^2 - 2 \frac{1+\gamma^2(1-2\alpha)}{1-\gamma^2} \operatorname{Re} \left\{ -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right\} + \left[\frac{1+\gamma^2(1-2\alpha)}{1-\gamma^2} \right]^2 < \left[\frac{1+\gamma^2(1-2\alpha)}{1-\gamma^2} \right]^2 + \frac{\gamma^2(2\alpha-1)^2-1}{1-\gamma^2},$$

that is

$$\left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} - \frac{1-\gamma^2(2\alpha-1)}{1-\gamma^2} \right| < \frac{2\gamma(1-\alpha)}{1-\gamma^2}. \quad (2.2)$$

The inequality (2.2) shows that the values region of the function $F(z) = -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)}$ is contained in the disc centered at $\frac{1-\gamma^2(2\alpha-1)}{1-\gamma^2}$ and radius $\frac{2\gamma(1-\alpha)}{1-\gamma^2}$.

The function $W = G(z) = \frac{1-(2\alpha-1)\gamma z}{1-\gamma z}$ maps the unit disc U onto the disc

$$|W - \frac{1-\gamma^2(2\alpha-1)}{1-\gamma^2}| < \frac{2\gamma(1-\alpha)}{1-\gamma^2}. \quad (2.3)$$

Since G is univalent and $F(0) = G(0)$, $F(U) \subset G(U)$, we obtain that $F(z) \prec G(z)$, that is

$$\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \prec \frac{1-(2\alpha-1)\gamma z}{1-\gamma z}$$

or

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \prec \frac{p(2\alpha-1)\gamma z - p}{1-\gamma z}$$

Conversely, suppose that

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \prec \frac{p(2\alpha-1)\gamma z - p}{1-\gamma z} \quad (z \in U). \quad (2.4)$$

But for $\gamma \neq 0$, the function $W = G(z) = \frac{1-(2\alpha-1)\gamma z}{1-\gamma z}$ maps the unit disc U onto the disc

$$|W - \frac{1-\gamma^2(2\alpha-1)}{1-\gamma^2}| < \frac{2\gamma(1-\alpha)}{1-\gamma^2} \quad (2.5)$$

Hence

$$\left| -\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} - \frac{1-\gamma^2(2\alpha-1)}{1-\gamma^2} \right| < \frac{2\gamma(1-\alpha)}{1-\gamma^2}. \quad (2.6)$$

After simple calculations, from (2.6) we obtain

$$\left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1 \right| < \gamma \left| \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 2\alpha - 1 \right|$$

Therefore, we complete the proof of Theorem 2.1.

In Theorem 2.1, putting $\gamma = 1$ gives the following corollary

Corollary 2.2. A function $f \in \Sigma_p$ is in the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$ if and only if

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \prec \frac{p(2\alpha-1)z - p}{1-z} \quad (z \in U). \quad (2.7)$$

Remark 2.3. Since

$$\operatorname{Re} \left[\frac{1-(2\alpha-1)\gamma z}{1-\gamma z} \right] > \alpha, \text{ it follows that } -\operatorname{Re} \left\{ \frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \right\} > \alpha,$$

which shows that $L_{p,\lambda,q,s}^n f(z) \in \Sigma_p^*(\alpha)$.

Using the subordination relationship for the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$, we derive a structural formula, first for the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$ and then for the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$.

Theorem 2.4. A function $f \in \Sigma_p$ belong to the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$ if and only if there exists a probability measure $\mu(x)$ on ∂U such that

$$f(z) = [z^{-p} + \sum_{k=1+p}^{\infty} \frac{z^k}{\rho_k(\alpha_1, \lambda, p, n)}] * [z^{-p} \exp(\int_{|x|=1} 2p(1-\alpha) \log(1-xz) d\mu(x))] \quad (z \in U^*). \quad (2.8)$$

Proof:

In view of the subordination condition (2.7) and Remark 2.3, we obtain that $f \in L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$ if and only if

$$-\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} \in P(\alpha)$$

From Limma 1.2, we have

$$-\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} = \int_{|x|=1} \frac{1-(2\alpha-1)xz}{1-xz} d\mu(x). \quad (2.9)$$

Integrating (2.9) with respect to z we get

$$L_{p,\lambda,q,s}^n f(z) = z^{-p} \exp(\int_{|x|=1} p(1-\alpha) \log(1-xz) d\mu(x)), \quad (2.10)$$

Now, from (1.16), (1.17) and (2.10) we obtain

$$f(z) = [z^{-p} + \sum_{k=1+p}^{\infty} \frac{1}{\rho_k(\alpha_1, p, \lambda, n)} z^k] * (f * 1)(z). \quad (2.11)$$

Hence

$$f(z) = [z^{-p} + \sum_{k=1+p}^{\infty} \frac{1}{\rho_k(\alpha_1, p, \lambda, n)} z^k] * [z^{-p} \exp(\int_{|x|=1} 2p(1-\alpha) \log(1-xz) d\mu(x))]. \quad (2.12)$$

Therefore, we complete the proof of Theorem 2.4.

Using a result of Goluzin [5] (see also [12]), we obtain the following

Theorem 2.5. Let a function $f \in \Sigma_p$ belong to the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$. Then $z^p L_{p,\lambda,q,s}^n f(z) < (1-z)^{2p(1-\alpha)}$ ($z \in U$).

Proof:

Let $f \in L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$. Then by Corollary 2.2 we have

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} < \frac{p(2\alpha-1)z-p}{1-z} \quad (z \in U).$$

The function $\frac{p(2\alpha-1)z-p}{1-z}$ is univalent and convex in U , then in view of Goluzin's result, we obtain

$$\int_0^z \frac{z(L_{p,\lambda,q,s}^n f(\zeta))'}{L_{p,\lambda,q,s}^n f(\zeta)} d\zeta < \int_0^z \frac{p(2\alpha-1)\zeta-p}{1-\zeta} d\zeta,$$

or

$$\log(L_{p,\lambda,q,s}^n f(z)) < \log \frac{(1-z)^{2p(1-\alpha)}}{z^p},$$

thus, there exists a function $w \in \Omega$ such that

$$\log(L_{p,\lambda,q,s}^n f(z)) = \log \frac{(1-w(z))^{2p(1-\alpha)}}{w(z)^p},$$

or

$$w(z)^p L_{p,\lambda,q,s}^n f(z) = (1-w(z))^{2p(1-\alpha)}$$

which is equivalent to the required result.

$$z^p L_{p,\lambda,q,s}^n f(z) < (1-z)^{2p(1-\alpha)}.$$

Hence, we complete the proof of Theorem 2.5

Next, we obtain a structural formula for the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$.

Theorem 2.6. Let a function $f \in \Sigma_p$ belong to the subclass $L_{p,\lambda,q,s}^n[\alpha_1, \alpha, 1]$. Then

$$f(z) = [z^{-p} + \sum_{k=1+p}^{\infty} \frac{z^k}{\rho_k(\alpha_1, \lambda, p, n)}] * [z^{-p} \exp(2p(1-\alpha) \gamma \int_0^z \frac{w(\zeta)}{\zeta(1-\zeta w(\zeta))} d\zeta)] \quad (z \in U^*). \quad (2.13)$$

where $w \in \Omega$.

Proof:

Suppose that $f \in L_{p,\lambda,q,s}^n[\alpha_1, \alpha, \gamma]$, then from Theorem 2.1 follows that

$$\frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} = \frac{p(2\alpha-1)\gamma w(z)-p}{1-\gamma w(z)} \quad (z \in U^*). \quad (2.14)$$

or

$$\frac{(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} = \frac{p(2\alpha-1)\gamma w(z)}{z(1-\gamma w(z))} - \frac{p\gamma w(z)+p-p\gamma w(z)}{z(1-\gamma w(z))} \quad (z \in U^*). \quad (2.15)$$

or

$$\frac{(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + \frac{p}{z} = \frac{2p(\alpha-1)\gamma w(z)}{z(1-\gamma w(z))} \quad (z \in U^*). \quad (2.16)$$

Taking integration of (2.16) with respect to z , we get

$$\log(z^p L_{p,\lambda,q,s}^n f(z)) = 2p(1-\alpha) \gamma \int_0^z \frac{w(\zeta)}{\zeta(1-\zeta w(\zeta))} d\zeta, \quad (2.17)$$

or

$$L_{p,\lambda,q,s}^n f(z) = z^{-p} \exp(2p(1-\alpha) \gamma \int_0^z \frac{w(\zeta)}{\zeta(1-\zeta w(\zeta))} d\zeta). \quad (2.18)$$

Now, from (1.16), (1.17) and (2.18) we get the result

$$f(z) = [z^{-p} + \sum_{k=1+p}^{\infty} \frac{z^k}{\rho_k(\alpha_1, p, \lambda, n)}] * [z^{-p} \exp(2p(1-\alpha)\gamma \int_0^z \frac{w(\zeta)}{\zeta(1-\zeta w(\zeta))} d\zeta)] \quad (z \in U^*). \quad (2.19)$$

Hence, we complete the proof of Theorem 2.6.

Theorem 2.7. Let $f \in \Sigma_p$ be given as in (1.6). If for α ($0 \leq \alpha < 1$) and γ ($0 < \gamma \leq 1$)

$$\sum_{k=1+p}^{\infty} [k(\gamma + 1) + p(1 + \gamma(2\alpha - 1))\rho_k(\alpha_1, p, \lambda, n)] |a_k| \leq 2p\gamma(1 - \alpha). \quad (2.20)$$

Then $f \in L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$.

Proof;

Since

$$f(z) = z^{-p} + \sum_{k=1+p}^{\infty} a_k z^k.$$

We have

$$\begin{aligned} M &= |z(L_{p,\lambda,q,s}^n f(z))'| + p |z(L_{p,\lambda,q,s}^n f(z))'| - \gamma |z(L_{p,\lambda,q,s}^n f(z))'| + p(2\alpha - 1) |L_{p,\lambda,q,s}^n f(z)| \\ &= |z(z^{-p} + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, p, \lambda, n) a_k z^k)'| + p |z(z^{-p} + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, p, \lambda, n) a_k z^k)| - \gamma \\ &\quad |z(z^{-p} + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, p, \lambda, n) a_k z^k)'| + p(2\alpha - 1) |z(z^{-p} + \sum_{k=1+p}^{\infty} \rho_k(\alpha_1, p, \lambda, n) a_k z^k)| \\ &= |\sum_{k=1+p}^{\infty} (k+p)\rho_k(\alpha_1, p, \lambda, n) a_k z^k| - \gamma |\frac{-2p(1-\alpha)}{z^p} + \sum_{k=1+p}^{\infty} (k+p(2\alpha-1))\rho_k(\alpha_1, p, \lambda, n) a_k z^k|. \end{aligned}$$

For $0 < |z| = r < 1$, we obtain

$$r^p M \leq \sum_{k=1+p}^{\infty} (k+p)\rho_k(\alpha_1, p, \lambda, n) |a_k| r^{p+k} - \gamma [2p(1-\alpha) - \sum_{k=1+p}^{\infty} (k+p(2\alpha-1))\rho_k(\alpha_1, p, \lambda, n) |a_k| r^{p+k}],$$

or

$$r^p M \leq \sum_{k=1+p}^{\infty} [k(\gamma + 1) + p(1 + \gamma(2\alpha - 1))\rho_k(\alpha_1, p, \lambda, n)] |a_k| r^{p+k} - 2p\gamma(1 - \alpha).$$

Since the above inequality holds for all r ($0 < r < 1$), letting $r \rightarrow 1$, we have

$$M \leq \sum_{k=1+p}^{\infty} [k(\gamma + 1) + p(1 + \gamma(2\alpha - 1))\rho_k(\alpha_1, p, \lambda, n)] |a_k| - 2p\gamma(1 - \alpha).$$

Making use of (2.20), we obtain $M < 0$, that is

$$|\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1| < \gamma |\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 2\alpha - 1|.$$

Consequently, $f \in L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$.

Therefore, we complete the proof of Theorem 2.7.

3. PROPERTIES OF THE SUBCLASS $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$

In this section we prove that the condition in (2.20) is both necessary and sufficient for a function to be in the subclass $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$

Theorem 3.1. Let $f \in \Sigma_p^+$. Then f belongs to the subclass $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$ if and only if

$$\sum_{k=1+p}^{\infty} [k(\gamma + 1) + p(1 + \gamma(2\alpha - 1))\rho_k(\alpha_1, p, \lambda, n)] |a_k| \leq 2p\gamma(1 - \alpha).$$

Proof;

In view of Theorem 2.7, we have to prove "only if" part.

Assume that $f(z) = z^{-p} + \sum_{k=1+p}^{\infty} a_k z^k$, $a_k \geq 0$ ($z \in U^*$), is in the subclass $L_{p,\lambda,q,s}^{n,+}[\alpha_1, \alpha, \gamma]$.

Then

$$\left| \frac{\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 1}{\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)} + 2\alpha - 1} \right| = \left| \frac{\sum_{k=1+p}^{\infty} (k+p)\rho_k(\alpha_1, p, \lambda, n) a_k z^k}{\frac{2p(1-\alpha)}{z^p} - \sum_{k=1+p}^{\infty} [k+p(2\alpha-1)]\rho_k(\alpha_1, p, \lambda, n) a_k z^k} \right| < \gamma,$$

for all $z \in U$. Since $\operatorname{Re}(z) \leq |z|$ for all z , it follows that

$$\operatorname{Re} \left\{ \frac{\sum_{k=1+p}^{\infty} (k+p)\rho_k(\alpha_1, p, \lambda, n) a_k z^k}{\frac{2p(1-\alpha)}{z^p} - \sum_{k=1+p}^{\infty} [k+p(2\alpha-1)]\rho_k(\alpha_1, p, \lambda, n) a_k z^k} \right\} < \gamma. \quad (3.1)$$

We choose the values of z on the real axis such that $\frac{1}{p} \frac{z(L_{p,\lambda,q,s}^n f(z))'}{L_{p,\lambda,q,s}^n f(z)}$ is real. Upon clearing the denominator in

(3.1) and letting $z \rightarrow 1$ through positive values, we obtain

$$\begin{aligned} \sum_{k=1+p}^{\infty} (k+p)\rho_k(\alpha_1, p, \lambda, n) a_k &\leq 2p\gamma(1-\alpha) \\ &\quad - \sum_{k=1+p}^{\infty} \gamma[k+p(2\alpha-1)]\rho_k(\alpha_1, p, \lambda, n) a_k, \end{aligned}$$

or

$$\sum_{k=1+p}^{\infty} [k(\gamma + 1) + p(1 + \gamma(2\alpha - 1))\rho_k(\alpha_1, p, \lambda, n)] |a_k| \leq 2p\gamma(1 - \alpha).$$

Hence, we complete the proof of Theorem 3.1.

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