



Optimal control of an N-policy two-phase $M^X/M/1$ queueing system with server startup subject to the server breakdowns and delayed repair

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Abstract: This paper investigates the economic behaviour of the two-phase $M^X/M/1$ queueing system with N-policy, server startup time and unreliable server, consisting of breakdown and delay periods. The service is in two phases of which the first phase is batch service provider to all customers waiting in the queue and the second phase is individual provider to each customer in the batch. The server is turned off and takes a vacation whenever the system is empty and turned on when the total number of customers in the system reaches the threshold N ($N \geq 1$), and starts preparatory work before providing the batch service. While the server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. There may be delay in repair due to non-availability of the repairing facility. The startup times, batch service times, individual service times, breakdown and delay periods are assumed to follow an exponential distribution. The closed form of expressions for the performance measures of interest is obtained. The mean queue waiting time by heuristic interpretation and the reliability indices of this model are also derived. The total expected cost function per unit time is developed to determine the optimal threshold of N at a minimum cost. The sensitivity analysis is presented through numerical illustration.

Keywords: Two-phase, vacation, breakdowns, N-policy, repair time, delay time, cost function.

I. Introduction.

A. Literature review

In most queueing systems the server may be subjected to lengthy and unpredictable breakdowns while serving a customer. For instance, in manufacturing systems the machine may breakdown due to malfunction or job related problems; in computer systems, the machine may be subject to scheduled backups and predicted failures. In these systems, server breakdown results in a period of server unavailability time until it is fixed. Therefore, it is necessary to see how the breakdowns affect the level of performance of the system.

Regarding the queueing systems with two phases of service Krishna and Lee [5] and Doshi[4] studied the distributed systems where all customers waiting in the queue receive batch service in the first phase of service followed by individual service in the second phase. Selvam and Sivasankaran [7] introduced the two phase queueing system with server vacations. Kim and Chae [6] investigated the two phase system with N-policy. The server startups correspond to the preparatory work of the server before starting the service. In some actual situations, the server often requires a startup time before starting each service period. Baker[1] first proposed the N-policy $M/M/1$ queueing system with exponential startup time. Several authors have investigated queueing models with server breakdowns and vacations in different frameworks in recent past. Wang [13] for the first time proposed the Markovian queueing system under the N-policy with server breakdowns. Vasanta Kumar and Chandan [8] presented the optimal operating policy for the two-phase $M^X/E_k/1$ queueing system under N-policy. Vasanta Kumar et al. [9,10]. presented the optimal control of $M^X/M/1$ and $M/E_k/1$ gated queueing systems with server startup and break downs. Vasanta Kumar et al.[11,12] presented the optimal operating policy for a two-phase $M^X/M/1$ and $M^X/E_k/1$ queueing systems under N-policy with server breakdowns and without gating. Choudhury and Tadj [2] considered an $M/G/1$ model with an additional phase of optional service with the assumption that the server is subject to breakdowns and delayed repair. Later Choudury et al. [3] investigated such a type of model, where the concept of N-policy is also introduced along with a delayed repair for batch arrival queueing system. In the two-phase $M^X/M/1$ queueing system without gating analyzed by Vasanta Kumar et al.[11] breakdowns are considered in second phase of service, without taking into consideration the concept of delayed repair.

Hence the present paper aims at the study of economic behavior of an N-policy $M^x/M/1$ queue in which service is in two-phases and the server is typically subjected to unpredictable breakdowns in both phases of service and delayed repair.

B. Model description and assumptions

We consider an $M^x/M/1$ queueing system with server startup, two-phases of service, unreliable server, and delay in repair due to non-availability of the service facility operating under N-policy.

Assumptions of the model

- (i). The arrival process is a compound Poisson process (with rate λ) of independent and identically distributed random batches of customers, where each batch size X , has a probability density function $\{a_n: a_n = P(X=n), n \geq 1\}$. Batches are admitted to service on a first come first served basis.
- (ii). The service is in two phases. The first phase of service is batch service to all customers waiting in the queue. On completion of batch service the server proceeds to second phase to serve all customers in the batch individually. Batch service time is assumed to be exponentially distributed with mean $1/\beta$ and is independent of batch size. Individual service times are also assumed to be exponentially distributed with mean $1/\mu$. On completion of individual service, the server returns to the batch queue to serve the customers who have arrived. If customers are waiting, the server starts the batch service followed by individual service to each customer in the batch. If no customer is waiting, the server takes a vacation.
- (iii). Whenever, the system becomes empty, the server is turned off. As soon as the total number of arrivals in the queue reaches to a predetermined threshold N the server is turned on and is temporarily unavailable for the waiting customers to restart service. It needs a startup time which follows an exponential distribution with mean $1/\theta$. As soon as the server finishes startup, it starts serving the first phase of waiting customers.
- (iv). The customers who arrive during the pre-service and batch service are also allowed to enter the same batch which is in service.
- (v). The server is subject to breakdowns at any time with Poisson breakdown rates α_1 for the first phase of service and α_2 for the second phase of service where it is working. Whenever, the server fails, it is sent for repair during which the server stops providing service and waits for the repair to get started. The waiting time for repair is defined as delay time and is assumed to be exponentially distributed with mean $1/\delta$. Repair time in any phase of service is assumed to be exponentially distributed with mean $1/\gamma$.
- (vi). In case the server breaks down while serving customers, it is sent for repair and that particular batch of customers or the customer who is just being served should wait for the server to come back to complete the remaining service. Immediately after the server is repaired, it starts to serve and the service time is cumulative. A customer who arrives and finds the server busy or broken down must wait in the queue until a server is available. Customers continue to arrive during the delay and repair periods of the broken server.

The primary objectives of the paper:

- (i) To establish the steady state equations and to obtain the steady state probability distribution of the number of customers present in the system.
- (ii) To derive diverse system characteristics such as expected number of customers in different states of the server: in vacation, in startup, in batch service, in individual service, in breakdown and delay states. An attempt is also made to estimate the expected length of the queue for the geometric batch size distribution.
- (iii) To obtain the mean queue waiting time by heuristic interpretation and reliability indices.
- (iv) Further, the paper aims to formulate the total expected cost function per unit time and determine the optimal value of the control parameter N .
- (v) Finally it proposes to perform a sensitivity analysis on the optimum value of N and the minimum expected cost through numerical experiments.

II. Steady state results

In steady state the following notations are used.

$P_{0,i,0}$ = The probability that there are i customers in the batch queue when server is on vacation, $i = 0, 1, 2, \dots$

$P_{1,i,0}$ = The probability that there are i customers in the batch queue when server is doing pre-service (startup work), where $i = N, N+1, N+2, \dots$

$P_{2,i,0}$ = The probability that there are i customers in the batch which is in batch service, $i = 1, 2, 3, \dots$

$P_{3,i,0}$ = The probability that there are i customers in the batch which is in batch service, but the server is found to be broken down and waiting for repair, $i = 1, 2, 3, \dots$

$P_{4,i,0}$ = The probability that there are i customers in the batch which is in batch service, but the server is under repair, $i = 1, 2, 3, \dots$

$P_{5,i,j}$ = The probability that there are i customers in the batch service queue and j customers in the individual service queue when the server is in individual service, $i = 0, 1, 2, \dots$, and $j = 1, 2, 3, \dots$

$P_{6,i,j}$ = The probability that there are i customers in the batch service queue and j customers in the individual service queue when the server is in individual service, but found to be broken down and waiting for repair, $i = 0, 1, 2, \dots$, and $j = 1, 2, 3, \dots$

$P_{7,i,j}$ = The probability that there are i customers in the batch service queue and j customers in the individual service queue when the server is in individual service, but the server is under repair, $i = 0, 1, 2, \dots$, and $j = 1, 2, 3, \dots$

The steady state equations satisfied by the system size probabilities are as follows:

$$\lambda P_{0,0,0} = \mu P_{5,0,1} \quad (2.1)$$

$$\lambda P_{0,i,0} = \lambda \sum_{l=1}^N a_l P_{0,i-l,0}, \quad 1 \leq i \leq N-1. \quad (2.2)$$

$$(\lambda + \theta) P_{1,N,0} = \lambda \sum_{l=1}^N a_l P_{0,N-l,0}, \quad i \geq 1. \quad (2.3)$$

$$(\lambda + \theta) P_{1,i,0} = \lambda \sum_{l=1}^{i-N} a_l P_{1,i-l,0} + \lambda \sum_{l=i-(N-1)}^i a_l P_{0,i-l,0}, \quad i = N+1, N+2, N+3, \dots \quad (2.4)$$

$$(\lambda + \beta + \alpha_1) P_{2,i,0} = \lambda \sum_{l=1}^i a_l P_{2,i-l,0} + \mu P_{5,i,1} + \gamma P_{4,i,0}, \quad 1 \leq i \leq N-1. \quad (2.5)$$

$$(\lambda + \beta + \alpha_1) P_{2,i,0} = \lambda \sum_{l=1}^i a_l P_{2,i-l,0} + \mu P_{5,i,1} + \gamma P_{4,i,0} + \theta P_{1,i,0}, \quad i \geq N. \quad (2.6)$$

$$(\lambda + \delta) P_{3,1,0} = \alpha_1 P_{2,1,0} \quad (2.7)$$

$$(\lambda + \delta) P_{3,i,0} = \alpha_1 P_{2,i,0} + \lambda \sum_{l=1}^i a_l P_{3,i-l,0}, \quad i > 1. \quad (2.8)$$

$$(\lambda + \gamma) P_{4,1,0} = \delta P_{3,1,0} \quad (2.9)$$

$$(\lambda + \gamma) P_{4,i,0} = \delta P_{3,i,0} + \lambda \sum_{l=1}^i a_l P_{4,i-l,0}, \quad i > 1. \quad (2.10)$$

$$(\lambda + \alpha_2 + \mu) P_{5,0,j} = \mu P_{5,0,j+1} + \beta P_{2,j,0} + \gamma P_{7,0,j}, \quad j \geq 1. \quad (2.11)$$

$$(\lambda + \alpha_2 + \mu) P_{5,i,j} = \mu P_{5,i,j+1} + \lambda \sum_{l=1}^i a_l P_{5,i-l,j} + \gamma P_{7,i,j}, \quad i \geq 1, j \geq 1. \quad (2.12)$$

$$(\lambda + \delta) P_{6,0,j} = \alpha_2 P_{5,0,j}, \quad j \geq 1. \quad (2.13)$$

$$(\lambda + \delta) P_{6,i,j} = \alpha_2 P_{5,i,j} + \lambda \sum_{l=1}^i a_l P_{6,i-l,j}, \quad i \geq 1, j \geq 1. \quad (2.14)$$

$$(\lambda + \gamma) P_{7,0,j} = \delta P_{6,0,j}, \quad j \geq 1. \quad (2.15)$$

$$(\lambda + \gamma) P_{7,i,j} = \delta P_{6,i,j} + \lambda \sum_{l=1}^i a_l P_{7,i-l,j}, \quad i \geq 1, j \geq 1. \quad (2.16)$$

Let $A(z) = \sum_{l=1}^{\infty} a_l z^l$ be the probability generating function of the arrival batch size random variable X

and $A^1(z)$, $A^1(z)$ represent the first and second order derivatives of $A(z)$ respectively.

The following generating functions are used to solve the equations (2.1) to (2.16)

$$G_q(z) = \sum_{i=0}^{\infty} P_{q,i,0} z^i, \quad 0 \leq q \leq 4, \text{ where } P_{0,i,0} = 0, \text{ if } i \geq N \text{ and } P_{1,i,0} = 0, \text{ if } i < N.$$

$$G_r(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{r,i,j} z^i y^j, \quad r = 5, 6, 7.$$

$$R_j(z) = \sum_{i=0}^{\infty} P_{5,i,j} z^i, \quad S_j(z) = \sum_{i=0}^{\infty} P_{6,i,j} z^i, \quad T_j(z) = \sum_{i=0}^{\infty} P_{7,i,j} z^i.$$

Solving equations (2.1) to (2.16) the following generating functions are obtained:

$$G_0(z) = P_{0,0,0} y_N(z), \text{ where } y_N(z) = \sum_{i=1}^{N-1} y_i z_i, \quad y_i = \sum_{l=1}^{N-1} a_l y_{i-l}, \quad y_0 = 1, \quad (2.17)$$

$$[\lambda(1 - A(z)) + \theta] G_1(z) = \lambda P_{0,0,0} + \lambda(A(z) - 1) G_0(z), \quad (2.18)$$

$$[\lambda(1 - A(z)) + \beta + \alpha_1] G_2(z) = \mu R_1(z) + \gamma G_4(z) + \theta G_1(z) - \lambda P_{0,0,0}, \quad (2.19)$$

$$[\lambda(1 - A(z)) + \delta] G_3(z) = \alpha_1 G_2(z), \quad (2.20)$$

$$[\lambda(1 - A(z)) + \gamma] G_4(z) = \delta G_3(z), \quad (2.21)$$

$$[\lambda y(1 - A(z)) + \alpha_2 y + \mu(y - 1)] G_5(z, y) = -\mu y R_1(z) + \gamma y G_7(z, y) + \beta y G_2(y), \quad (2.22)$$

$$[\lambda(1 - A(z)) + \delta] G_6(z, y) = \alpha_2 G_5(z, y), \quad (2.23)$$

$$[\lambda(1 - A(z)) + \gamma] G_7(z, y) = \delta G_6(z, y). \quad (2.24)$$

The total probability generating function $G(z, y)$ is given by

$$G(z, y) = G_0(z) + G_1(z) + G_2(z) + G_3(z) + G_4(z) + G_5(z, y) + G_6(z, y) + G_7(z, y). \quad (2.25)$$

The normalizing condition is

$$G(1, 1) = G_0(1) + G_1(1) + G_2(1) + G_3(1) + G_4(1) + G_5(1, 1) + G_6(1, 1) + G_7(1, 1) = 1. \quad (2.26)$$

This condition yields, $R_1(1) = \lambda A^1(1) / \mu$.

Under steady state conditions, let $P_v, P_s, P_b, P_{bb}, P_{db}, P_i, P_{bi}$ and P_{di} be the probabilities that the server is in vacation, in startup, in batch service, waiting for repair during batch service, under repair during batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively. Then

$$P_v = G_0(1) = y_N(1) P_{0,0,0}, \quad (2.27)$$

$$P_s = G_1(1) = (\lambda P_{0,0,0} / \theta), \quad (2.28)$$

$$P_b = G_2(1) = (\lambda A^1(1) / \beta), \quad (2.29)$$

$$P_{bb} = G_3(1) = (\alpha_1 / \delta) (\lambda A^1(1) / \beta), \quad (2.30)$$

$$P_{db} = G_4(1) = (\alpha_1 / \gamma) (\lambda A^1(1) / \beta), \quad (2.31)$$

$$P_i = G_5(1, 1) = (\lambda A^1(1) / \mu), \quad (2.32)$$

$$P_{bi} = G_6(1, 1) = (\alpha_2 / \delta) (\lambda A^1(1) / \mu), \quad (2.33)$$

$$P_{di} = G_7(1, 1) = (\alpha_2 / \gamma) (\lambda A^1(1) / \mu), \quad (2.34)$$

where $P_{0,0,0} = (1 - \rho)$, and $\rho = \frac{\lambda A^1(1)}{\beta} \left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta} \right) + \frac{\lambda A^1(1)}{\mu} \left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta} \right)$ is the

utilizing factor of the system.

A. Expected number of customers in different states

Using the probability generating functions expected number of customers in the system at different states are derived in this section.

Let $L_v, L_s, L_b, L_{bb}, L_{db}, L_i, L_{bi}$ and L_{di} be the expected number of customers in the system when the server is in vacation, in startup, in batch service, waiting for repair during batch service, under repair during batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively.

Differentiating the generating functions $G_q(z)$, $q = 0, 1, 2, 3, 4, 5, 6, 7$ given in equations (2.17) to (2.24) and substituting $z=1$ we get

$$L_v = G_0'(1) = y_N'(1) P_{0,0,0}, \quad (2.35)$$

$$L_s = G_1'(1) = [\lambda A'(1) (\lambda + \theta y_N(1)) / \theta^2] P_{0,0,0}, \quad (2.36)$$

$$L_b = G_2'(1) = \lambda A'(1) / \beta, \quad (2.37)$$

$$L_{bb} = G_3'(1) = \left(\frac{\lambda \alpha_1 A'(1)}{\beta \delta} \right) \left(1 + \frac{\lambda A'(1)}{\delta} \right), \quad (2.38)$$

$$L_{db} = \sum_{i=1}^{\infty} i P_{4,i,0} = G_4'(1) = \frac{\lambda \alpha_1 A'(1)}{\gamma \beta} \left(1 + \frac{\lambda A'(1)}{\gamma} + \frac{\lambda A'(1)}{\delta} \right), \quad (2.39)$$

$$L_i = G_5'(1, 1)$$

$$\begin{aligned} &= \frac{\lambda A'(1)}{\mu} \left[\frac{1}{(1-\rho_2)} + \frac{\rho_1}{(1-\rho_2)} + \frac{\lambda A'(1)}{(1-\rho_2)} \left(\frac{\lambda + \theta y_N(1)}{\theta^2} \right) P_{0,0,0} + \frac{\lambda A'(1) \rho_2}{(1-\rho_2)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) \right. \\ &+ \frac{\lambda A''(1)}{2 A'(1)(1-\rho_2)} \left(\frac{\lambda + \theta y_N(1)}{\theta} \right) P_{0,0,0} + \frac{y_N'(1)}{(1-\rho_2)} P_{0,0,0} + \frac{(\lambda A'(1))^2 \alpha_1}{\beta \gamma \delta (1-\rho_2)} \left(1 + \frac{\delta}{\gamma} + \frac{\gamma}{\delta} \right) \\ &\left. + \frac{A''(1)(\rho_1 + \rho_2)}{2 A'(1)(1-\rho_2)} - \frac{(\lambda A'(1))^2}{\mu (1-\rho_2)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) - \frac{(\lambda A'(1))^2 \alpha_2}{\mu \gamma \delta (1-\rho_2)} \right], \end{aligned} \quad (2.40)$$

$$L_{bi} = G_6'(1, 1) = \frac{(\lambda A'(1))^2 \alpha_2}{\mu \delta^2} + \frac{\alpha_2}{\delta} G_5'(1, 1), \quad (2.41)$$

$$L_{di} = G_7'(1, 1) = \frac{(\lambda A'(1))^2 \alpha_2}{\mu \gamma} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) + \frac{\delta}{\gamma} G_5'(1, 1), \quad (2.42)$$

where $\rho_1 = \frac{\lambda A'(1)}{\beta} \left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta} \right)$, $\rho_2 = \frac{\lambda A'(1)}{\mu} \left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta} \right)$ and

$G_q'(\cdot)$ denotes the first order derivative of $G_q(\cdot)$.

Finally, the expected number of units in the system

$$L(N) = L_v + L_s + L_b + L_{bb} + L_{db} + L_i + L_{bi} + L_{di}$$

$$\begin{aligned} &= \frac{\rho}{(1-\rho_2)} + \frac{y_N'(1)}{(1-\rho_2)} P_{0,0,0} + \frac{\lambda A'(1)}{(1-\rho_2)} \left(\frac{\lambda + \theta y_N(1)}{\theta^2} \right) P_{0,0,0} \\ &+ \frac{(\lambda A'(1))^2 \alpha_1}{\beta \gamma \delta (1-\rho_2)} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) + \frac{(\lambda A'(1))^2 \alpha_2}{\mu \gamma \delta} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) \\ &+ \frac{\lambda A'(1) \rho_2}{(1-\rho_2)} \left(\frac{1}{\gamma} + \frac{1}{\delta} \right) \left(\rho_2 - \frac{\lambda A'(1)}{\mu} \right) - \frac{(\lambda A'(1))^2 \alpha_2 \rho_2}{\mu \gamma \delta (1-\rho_2)} \\ &+ \frac{\lambda A''(1) \rho_2}{2 A'(1)(1-\rho_2)} \left(\frac{\lambda + \theta y_N(1)}{\theta} \right) P_{0,0,0} + \frac{A''(1) \rho_2 \rho}{2 A'(1)(1-\rho_2)}. \end{aligned} \quad (2.43)$$

III. Some other system performance measures

In this section, expected length of vacation period, startup period, batch service period, delay period during batch service, waiting period for repair during batch service, individual service period, delay period during individual service and waiting period for repair during individual service are presented.

The expected length of a busy cycle E_c is given by

$$E_c = E_v + E_s + E_b + E_{bb} + E_{db} + E_i + E_{bi} + E_{di} \quad (3.1)$$

The long run fractions of time the server is in different states are respectively,

$$E_v / E_c = P_v = y_N(1) P_{0,0,0} \quad (3.2)$$

$$E_s / E_c = P_s = (\lambda / \theta) P_{0,0,0} \quad (3.3)$$

$$E_b / E_c = P_b = \lambda A^1(1) / \beta \quad (3.4)$$

$$E_{bb} / E_c = P_{bb} = (\alpha_1 / \delta) \left(\frac{\lambda A^1(1)}{\beta} \right) \quad (3.5)$$

$$E_{db} / E_c = P_{db} = (\alpha_1 / \gamma) \left(\frac{\lambda A^1(1)}{\beta} \right) \quad (3.6)$$

$$E_i / E_c = P_i = \lambda A^1(1) / \mu \quad (3.7)$$

$$E_{bi} / E_c = P_{bi} = (\alpha_2 / \delta) \left(\frac{\lambda A^1(1)}{\mu} \right) \quad (3.8)$$

$$E_{di} / E_c = P_{di} = (\alpha_2 / \gamma) \left(\frac{\lambda A^1(1)}{\mu} \right) \quad (3.9)$$

Expected length of vacation period is given by

$$E_v = N / \lambda \quad (3.10)$$

Substituting this in equation (3.2)

$$E_c = 1 / (\lambda P_{0,0,0}) \quad (3.11)$$

A. Heuristic approach to waiting time in the queue

Let W_q be the waiting time of the test customer until his individual service. An arbitrary customer waits different time amounts according to the state of his arriving epoch. First, we divide the regeneration cycle into eight parts of the idle period, the startup period, the first phase batch service period, waiting time for repair and repair period due to breakdown in first phase, the second phase individual service period, waiting time for repair and repair period due to breakdown in second phase, and the repair period with respective probabilities

$$y_N(1)P_{0,0,0}, \frac{\lambda P_{0,0,0}}{\theta}, \frac{\lambda A^1(1)}{\beta}, \frac{\alpha_1}{\delta} \left(\frac{\lambda A^1(1)}{\beta} \right), \frac{\alpha_1}{\gamma} \left(\frac{\lambda A^1(1)}{\beta} \right), \frac{\lambda A^1(1)}{\mu}, \frac{\alpha_2}{\delta} \left(\frac{\lambda A^1(1)}{\mu} \right) \text{ and } \frac{\alpha_2}{\gamma} \left(\frac{\lambda A^1(1)}{\mu} \right).$$

That is the system state that the arriving customer sees determines his waiting time. The test customer has to wait during the individual service times for those already waiting (except the ongoing individual service) in the system. In addition to it,

- (i) If the server is idle, the customer has to wait the remaining idle period, startup period, the first phase batch service period.
- (ii) If the server is in the startup state, the customer has to wait the remaining startup period, the first phase batch service period.
- (iii) If the server is in the first phase, the customer has to wait the remaining time of the ongoing batch service.
- (iv) If the server is in the breakdown waiting state of batch service, the customer has to wait the remaining waiting period and repair period.
- (v) If the server is in the repair state due to breakdown during batch service, the customer has to wait the remaining repair period.
- (vi) If the server is in the second phase, the customer has to wait the remaining time of the ongoing individual service plus the batch service.

- (vii) If the server is waiting for repair due to breakdown during individual service, the customer has to wait the remaining waiting period and the repair period, the first phase batch service period.
 (viii) If the server is in the repair state due to breakdown during individual service, the customer has to wait the remaining repair time period plus the first phase batch service period.

Thus

$$\begin{aligned}
 E(W_q) &= L(N-1) \frac{1}{\mu} + \left(\frac{N-1}{2\lambda A'(1)} + \frac{1}{\theta} + \frac{1}{\beta} \right) y_N(1) P_{0,0,0} + \left(\frac{1}{\theta} + \frac{1}{\beta} \right) \frac{\lambda P_{0,0,0}}{\theta} + \left(\frac{\lambda A'(1)}{\beta^2} \right) \\
 &\quad + \left(\frac{1}{\delta} + \frac{1}{\gamma} \right) \left(\frac{\alpha_1}{\delta} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{\alpha_1}{\gamma^2} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{\lambda A'(1)}{\mu} \right) \left(\frac{1}{\mu} + \frac{1}{\beta} \right) \\
 &\quad + \left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{1}{\beta} \right) \left(\frac{\alpha_2}{\delta} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{1}{\gamma} + \frac{1}{\beta} \right) \left(\frac{\alpha_2}{\gamma} \right) \left(\frac{\lambda A'(1)}{\mu} \right). \\
 &= \frac{1}{1-r} \left[\left(\frac{N-1}{2\lambda A'(1)} + \frac{1}{\theta} + \frac{1}{\beta} \right) y_N(1) P_{0,0,0} + \left(\frac{1}{\theta} + \frac{1}{\beta} \right) \left(\frac{\lambda P_{0,0,0}}{\theta} \right) + \left(\frac{\lambda A'(1)}{\beta^2} \right) \right. \\
 &\quad + \left(\frac{1}{\delta} + \frac{1}{\gamma} \right) \left(\frac{\alpha_1}{\delta} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{\alpha_1}{\gamma^2} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{\lambda A'(1)}{\mu} \right) \left(\frac{1}{\mu} + \frac{1}{\beta} \right) \\
 &\quad \left. + \left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{1}{\beta} \right) \left(\frac{\alpha_2}{\delta} \right) \left(\frac{\lambda A'(1)}{\beta} \right) + \left(\frac{1}{\gamma} + \frac{1}{\beta} \right) \left(\frac{\alpha_2}{\gamma} \right) \left(\frac{\lambda A'(1)}{\mu} \right) \right].
 \end{aligned}$$

$$\text{where } r = \left(\frac{\lambda A'(1)}{\mu} \right) - \left(\frac{\lambda A'(1)}{\beta} \right).$$

B. Reliability indices

In this section two reliability indices of the system viz.- The system availability and failure frequency under the steady state conditions are discussed. Let $A_v(t)$ be the system availability at time t , that is the probability that the server is working for a customer or in idle period or in startup period, or in batch service such that the steady state availability of the server will be

$$A_v = \lim_{t \rightarrow \infty} A_v(t)$$

The steady state availability of the server will be given by

$$\begin{aligned}
 &G_0(1) + G_1(1) + G_2(1) + G_3(1,1) \\
 &= \frac{(\lambda + y_N(1)\theta)}{\theta} P_{0,0,0} + \frac{\lambda A'(1)}{\mu} + \frac{\lambda A'(1)}{\mu} \\
 &= 1 - \frac{\lambda A'(1)}{\beta} \left(\frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta} \right) - \frac{\lambda A'(1)}{\mu} \left(\frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta} \right)
 \end{aligned}$$

The steady state failure frequency of the server is given by

$$\frac{\lambda \alpha_1 A'(1)}{\beta} + \frac{\lambda \alpha_2 A'(1)}{\mu}.$$

IV. Optimal Cost Structure

In this section, the optimal value of N is determined that minimizes the long run average cost of two-phase $M^x/M/1$, N -policy queue with server break downs and delay in repair. To determine the optimal value of N the following linear cost structure is considered.

Let $C_A(N)$ be the average cost per unit of time, then

$$C_A(N) = C_h L(N) + C_0 \left(\frac{E_b}{E_c} + \frac{E_i}{E_c} \right) + C_m \left(\frac{E_s}{E_c} \right) + C_b \left(\frac{E_{bb} + E_{db} + E_{bi} + E_{di}}{E_c} \right) + C_s \left(\frac{1}{E_c} \right) - C_r \left(\frac{E_v}{E_c} \right). \quad (4.1)$$

where

$C_h \equiv$ Holding cost per unit of time spend for each customer present in the system,

$C_0 \equiv$ Cost per unit of time for keeping the server on and in operation,

$C_m \equiv$ Start up cost per unit time,

$C_s \equiv$ Setup cost per cycle,

$C_b \equiv$ Break down cost per unit of time for the unreliable server, and

$C_r \equiv$ Reward per unit of time as the server is doing secondary work during vacation.

From equations (3.4) to (3.9) it is observed that

E_b / E_c , E_{bb} / E_c , E_{db} / E_c , E_i / E_c , E_{bi} / E_c , E_{di} / E_c are not functions of the decision variable N .

Hence, for determination of the optimal operating N -policy, minimizing $C_A(N)$ in equation (4.1) is equivalent to minimizing

$$T_A(N) = \frac{(1-\rho) \left[C_h Y_N^{-1}(1) + (1-\rho_2) \left\{ C_m \frac{\lambda}{\theta} + \lambda C_s - C_r Y_N(1) \right\} \right]}{(1-\rho_2)(Y_N(1) + (\lambda/\theta))}. \quad (4.2)$$

It is hard to prove that $T_A(N)$ is convex. But a procedure that makes it possible to calculate the optimal threshold N^* is presented below.

V. Result

Under the long run expected average cost criterion, the optimal threshold N^* for the model is the best value of 'k' given by

$$N^* = \min. \left\{ k \geq 1 / \sum_{j=0}^N (k-j) y_j + \frac{k\lambda}{\theta} > \frac{\lambda(1-\rho_2)}{C_h} \left(\frac{C_m - C_r}{\theta} + C_s \right) \right\} \quad (4.3)$$

and it is one of the integers surrounding N .

Proof: See ref. (11).

Sensitivity analysis

In this section, sensitivity analysis is performed on the optimum threshold N^* based on changes in the system parameters and the cost elements numerical illustrations.

Let the batch size X have geometric distribution with mean batch size $1/p$. Then $a_j = P(X=j) = p(1-p)^{j-1}$, $0 < p < 1$, $j=1,2,3,\dots$ with the probability generating function

$$A(z) = pz / [1 - (1-p)z] \text{ and } E(X) = A^1(1) = 1/p,$$

$$E[X(X-1)] = A^{11}(1) = 2(1-p)/p^2.$$

The expected number of customers in the system is given by

$$L(N) = \frac{\rho}{(1-\rho_2)} + \frac{\sum_{i=0}^{N-1} y_i P_{0,0,0}}{(1-\rho_2)} + \frac{\lambda}{p(1-\rho_2)} \left(\frac{\lambda + \sum_{i=0}^{N-1} y_i}{\theta^2} \right) P_{0,0,0} \\ + \frac{\lambda^2 \alpha_1}{p^2 \beta \gamma \delta (1-\rho_2)} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) + \frac{\lambda^2 \alpha_2}{p^2 \mu \gamma \delta} \left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma} \right) \\ + \frac{\lambda(1-p)\rho_2}{p(1-\rho_2)} \left(\frac{\lambda + \sum_{i=0}^{N-1} y_i}{\theta} \right) P_{0,0,0} + \left(\frac{1-p}{p} \right) \frac{\rho_2 \rho}{(1-\rho_2)} \\ + \frac{\lambda \rho_2}{p(1-\rho_2)} \left(\frac{1}{\delta} + \frac{1}{\gamma} \right) \left(\rho_2 - \frac{\lambda}{p\mu} \right) - \frac{\lambda^2 \alpha_2 \rho_2}{p^2 \mu \gamma \delta (1-\rho_2)}$$

where $y_i = \sum_{l=1}^{N-1} y_{i-1}$, $y_0 = 1$, $a_j = p(1-p)^{j-1}$, $\rho_1 = \frac{\lambda}{p\beta} \left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta} \right)$, $\rho_2 = \frac{\lambda}{p\mu} \left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta} \right)$,
 $P_{0,0,0} = (1 - \rho_1 - \rho_2) \frac{\theta}{\left(\lambda + \theta \sum_{i=0}^{N-1} y_i \right)}$.

The varying details of optimal threshold N^* and the minimum cost $C_A(N^*)$ for specified values of the system parameters and the cost elements are presented in the following tables.

Table1: The optimal N^* and minimum expected cost $C_A(N^*)$ with various (λ, γ)

$\beta=4, \theta=3, \gamma=3, m=2, \alpha_1=0.2, \alpha_2=0.5, \delta=2, C_h=5, C_0=50, C_m=100, C_b=100, C_r=40, C_s=1000$					
(λ, μ)	N^*	$C_A(N^*)$	(λ, μ)	N^*	$C_A(N^*)$
(0.2,3.5)	11	30.78	(0.4,2.0)	11	66.32
(0.3,3.5)	13	47.04	(0.4,2.5)	12	62.28
0.4,3.5)	14	58.31	(0.4,3.0)	13	59.92
(0.5,3.5)	15	65.32	(0.4,3.5)	14	58.31
(0.6,3.5)	15	68.15	(0.4,4.0)	14	57.14

From Table 1 it may be observed that (a) N^* shows increasing trend for increase in the values of λ and μ , (b) $C_A(N^*)$ increases with increase in λ and decreases with increase in μ .

Table2: The optimal N^* and minimum expected cost $C_A(N^*)$ with various (β, θ)

$\lambda=0.4, \mu=3.5, \beta=4, \theta=3, \gamma=3, m=2, \alpha_1=0.2, \alpha_2=0.5, \delta=2, C_h=5, C_0=50, C_m=100, C_b=100, C_r=40, C_s=1000$					
(β, θ)	N^*	$C_A(N^*)$	(β, θ)	N^*	$C_A(N^*)$
(3.0,3.0)	14	58.71	(4.0,2.0)	14	58.90
(4.0,3.0)	14	58.31	(4.0,2.5)	14	58.55
(5.0,3.0)	14	58.07	(4.0,3.0)	14	58.31
(6.0,3.0)	14	57.91	(4.0,3.5)	14	58.14
(7.0,3.0)	14	57.80	(4.0,4.0)	14	58.01

It can be seen from Table 2 that (a) N^* is insensitive to the values of β and θ , (b) with increase in the values of β , $C_A(N^*)$ decreases. Conversely it decreases with increase in the values of θ .

Table 3: The optimal N^* and minimum expected cost $C_A(N^*)$ with various (c_h, c_0)

$\lambda=0.4, \mu=3.5, \beta=4, \theta=3, \gamma=3, m=2, \alpha_1=0.2, \alpha_2=0.2, \delta=2, C_m=100, C_b=100, C_r=40, C_s=1000$					
(c_h, c_0)	N^*	$C_A(N^*)$	(c_h, c_0)	N^*	$C_A(N^*)$
(4.0,50)	16	52.71	(5.0,40)	14	54.02
(6.0,50)	13	63.44	(5.0,60)	14	62.60
(8.0,50)	11	72.62	(5.0,80)	14	71.17
(10,50)	10	80.88	(5.0,100)	14	79.74
(12,50)	9	88.41	(5.0,120)	14	88.31

It can be observed from table 3 that (a) N^* is insensitive to increase in C_0 where as it decreases with increase in C_h and (b) $C_A(N^*)$ increases with increase in the values of C_h and C_0 .

Also, it is observed from the numerical computations (which are not presented) that (a) N^* is insensitive to increase in the values of γ , δ , α_1 , α_2 , C_m and C_r (b) $C_A(N^*)$ decreases with increase in the values of γ , δ and C_r and increases with increase in the values of α_1 , α_2 and C_m .

VI. Conclusions

In this paper, some important performance measures are derived for the N-policy, $M^x/M/1$ queueing system with two phases of service, server startup, server breakdowns and delayed repair. As the convexity of the expected cost function cannot be proved theoretically, a heuristic approach is chosen to determine the optimal threshold. Sensitivity analysis is performed between the optimal value of N , the specific system parameters and cost elements for the specific batch size distribution, geometric.

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